Selected Topics in PDE part 5.

John Andersson johnan@kth.se

November 3, 2014

Contents

1 Variable coefficients. **1**
1.1 The maximum Principle for Elliptic PDE. 3

2 Apriori estimates. **7**
2.1 Discussion. 7
2.2 Interior Aproiri Estimates for the Laplacian. 8
2.2.1 An application. 16

Chapter 1

Variable coefficients.

So far we have been able to show existence for solutions to the Dirichlet problem for Laplace equation. It is of some interest to generalize that result to more general equations. We will consider the following general elliptic second order PDE,

$$\begin{array}{ll} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array} \quad (1.1)$$

where Ω is some bounded domain, $g \in C(\partial\Omega)$, $a_{ij}(x)$, $b_i(x)$, $c(x)$ and $f(x)$ are given functions.

Equation (1.1) is to general for us to be able to say anything specific about the solution $u(x)$. We need to impose some conditions on $a_{ij}(x)$, $b_i(x)$ and $c(x)$ to assure that the solutions are "well behaved".

A powerful tool we used in the solution of the Laplace equation was the maximum principle. To assure that solutions $u(x)$ to (1.1) satisfy the maximum principle we make the following definition.

Definition 1. *We say that an partial differential, equation defined a domain* Ω,

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x)$$

is strictly elliptic in Ω *if there exists a constant* $\lambda > 0$ *such that for all* $x \in \Omega$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for any vector $\xi = (\xi_1, \xi_2, ..., \xi_n)$.

Remark: If we let A be the matrix with coefficients $a_{ij}(x)$ then the ellipticity condition say that

$$\begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & & a_{nn}(x) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \geq \lambda |\xi|^2.$$

This is the same as demanding that all (generalized) eigenvalues[1] of A are greater than λ.

One might ask what ellipticity has to do with the maximum principle. A simple example will suffice to show that ellipticity is related to the maximum principle.

Example: Let Ω be a bounded domain, $\epsilon > 0$ and $u(x) \in C^2(\Omega) \cap C(\Omega)$ be a solution to

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x) = \epsilon \quad \text{in } \Omega.$$

Assume furthermore, for simplicity, that $a_{ii}(x) = a_i(x)$ and $a_{ij}(x) = 0$ for $i \neq j$, $a_{ii}(x) \geq 1$ (that is the PDE is elliptic with $\lambda = 1$) and that $c(x) \leq 0$. Then $u(x)$ does not have any non-negative interior maximum.

This is quite obvious. We argue by contradiction and assume that $u(x)$ has an interior non-negative maximum at $x^0 \in \Omega$. Then $\nabla u(x^0) = 0$ and $\frac{\partial^2 u(x^0)}{\partial x_i^2} \leq 0$. We can thus calculate

$$0 < \epsilon = \underbrace{\sum_{i=1}^{n} a_{ii}(x^0) \frac{\partial^2 u(x^0)}{\partial x_i^2}}_{\leq 0} + \underbrace{\sum_{i=1}^{n} b_i(x^0) \frac{\partial u(x^0)}{\partial x_i}}_{=0 \text{ since } \nabla u = 0} + \underbrace{c(x^0)u(x^0)}_{\leq 0} \leq 0,$$

where we used that $a_{ii}(x^0) \geq 1$, and $c(x) \leq 0$ by assumption and that $u(x^0) \geq 0$ since x^0 is the non-negative maximum. Clearly this is a contradiction. In particular, elliptic PDE with $c(x) \leq 0$ seems to satisfy a maximum principle.

Remark on different kinds of PDE: We will only study elliptic PDE in this course. However, there are other classes of important PDE that appears in the applied sciences. Besides elliptic the most important classes of PDE are *parabolic* and *hyperbolic*.

The heat equation,

$$\Delta u(x,t) - \frac{\partial u(x,t)}{\partial t} = 0,$$

is the archetypical *parabolic* equation. A parabolic equation is, more or less, an elliptic equation minus a time derivative.

[1] Since $\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \frac{\partial^2 u(x)}{\partial x_i \partial x_j}$ for a C^2 function there is no loss of generality to assume that A is diagonalizable and thus that the n eigenvalues exists.

The third important class of equations is represented by the *wave equation*

$$\Delta u(x,t) - \frac{\partial^2 u(x,t)}{\partial t^2} = 0.$$

The wave equation is the basic representative of the *hyperbolic* PDE.

Of the three classes of PDE one can say that elliptic and parabolic are the most similar. Most of the results for elliptic PDE also exist for parabolic PDE. For instance the maximum principle (suitably interpreted) and the regularity theory that we develop also exist for parabolic PDE. However, one needs to formulate the problems and results slightly different for parabolic PDE since the PDE has a time variable t. We will not discuss parabolic or hyperbolic equations in this course.

1.1 The maximum Principle for Elliptic PDE.

For simplicity we will write, for any $u \in C^2(\Omega)$

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) \quad \text{in } \Omega, \tag{1.2}$$

where L is an elliptic operator, $a_{ij}(x), b_i(x), c(x) \in C(\Omega)$.

Lemma 1. [The weak maximum principle.] *Suppose that $u \in C^2(\Omega)$, where Ω is a bounded domain, and $Lu(x) = f(x)$ where $f(x) \in C(\Omega)$. Assume that*

1. *$c(x) \leq 0$ and $f(x) > 0$ or*
2. *$c(x) < 0$ and $f(x) \geq 0$.*

The $u(x)$ does not achieve a positive local maximum in Ω.

In particular, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ then

$$\sup_{\Omega} u(x) = \sup_{\partial\Omega} u(x).$$

Proof: The proof is very similar to the example in the previous section. We argue by contradiction and assume that $u(x^0) > 0$ and that x^0 is a local maximum for $u(x)$. Then

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} =$$

$$= \underbrace{-c(x)u(x) + f(x)}_{>0 \text{ at } x^0}.$$

Since $u(x^0)$ is a local maximum we can conclude that $\nabla u(x^0) = 0$ and $D^2 u(x^0)$ is a non-positive matrix.

In particular,

$$0 < \sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} + \underbrace{\sum_{i=1}^{n} b_i(x^0)\frac{\partial u(x^0)}{\partial x_i}}_{=0} = \sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} \tag{1.3}$$

If we can show that the right hand side in (1.3) is non positive we get the desired contradiction.

Since the matrix $A(x^0) = [a_{ij}(x^0)]_{ij}$ is strictly positive by ellipticity it has a square root $\sqrt{A}$. Also $-D^2 u_\epsilon(x^0)$ is non-negative so it has a square root $\sqrt{-D^2 u(x^0)}$. Now we notice that

$$\sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} = \text{trace}\left(A \cdot D^2 u(x^0)\right) =$$

$$= -\text{trace}\left(A \cdot \left(-D^2 u(x^0)\right)\right) = -\text{trace}\left(\sqrt{A}\sqrt{A}\sqrt{-D^2 u(x^0)}\sqrt{-D^2 u(x^0)}\right) =$$

$$= -\text{trace}\left(\left(\sqrt{A}\sqrt{-D^2 u(x^0)}\right)^T \sqrt{A}\sqrt{-D^2 u(x^0)}\right) \leq 0,$$

where we have used linear algebra freely and that the last inequality follows from $\text{trace}(C^T \cdot C) = \sum_{i,j=1}^{n}(c_{ij})^2 \geq 0$ for any matrix C. This finishes the proof. □

Corollary 1. [THE COMPARISON PRINCIPLE.] *Let Ω be a bounded domain and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$\begin{aligned} Lu(x) &\geq Lv(x) \quad &&\text{in } \Omega \\ u(x) &\leq v(x) \quad &&\text{on } \partial\Omega. \end{aligned}$$

Then, if $c(x) \leq 0$, it follows that $u(x) \leq v(x)$ in Ω.

Proof: The proof is simple, and follows directly from Lemma 1 if $Lu(x) > Lv(x)$ since then $L(u - v) > 0$ and can not archive a positive maximum.

We will modify the function $u - v$ by a function w to obtain the strict inequality and then use Lemma 1 to prove the Corollary.

To that end we define

$$w(x) = e^{Nr^2} - e^{N|x|^2},$$

where r is chosen large enough that $w(x) \geq 0$ in Ω and N is to be determined later. Notice that

$$Lw(x) = \sum_{i,j=1}^{n} a_{ij}(x)\left(-2N\delta_{ij} - 4N^2 x_i x_j\right) e^{N|x|^2} + \tag{1.4}$$

$$+ \underbrace{\sum_{i=1}^{n} b_i(x)\left(-2N x_i\right) e^{N|x|^2}}_{\leq 2Ne^{N|x|^2}|x| \sup_\Omega |b(x)|} + \underbrace{c(x)(e^{Nr^2} - e^{N|x|^2})}_{\leq 0} \leq$$

$$\leq \sum_{i,j=1}^{n} a_{ij}(x)\left(-2N\delta_{ij} - 4N^2 x_i x_j\right) e^{N|x|^2} + 2Ne^{N|x|^2}|x| \sup_{\Omega} |b(x)|,$$

where we used that $c(x) \leq 0$ and $w \geq 0$ in Ω. We need to estimate

$$\sum_{i,j=1}^{n} a_{ij}(x)\left(-2N\delta_{ij} - 4N^2 x_i x_j\right) = -2N\sum_{i=1}^{n} a_{ii}(x) - 4N^2 \sum_{i,j=1}^{n} a_{ij}(x) x_i x_j \leq$$
$$\leq -2N\lambda - 4N^2\lambda|x|^2, \tag{1.5}$$

since the first sum is just the trace of A and the second sum can be estimated from below by $\lambda|x|^2$ by the definition of ellipticity with $\xi = x$.

Using (1.5) in the estimate (1.4) we can conclude that

$$Lw(x) \leq -4N^2\lambda\left(\frac{1}{2N} + |x|^2 - \frac{|x|\sup_\Omega |b(x)|}{2N\lambda}\right) e^{N|x|^2} =$$

$$= -4N^2\lambda\left(\underbrace{\left(|x| - \frac{B}{4N\lambda}\right)^2}_{\geq 0} + \frac{1}{2N} - \frac{1}{N^2}\left(\frac{B}{4\lambda}\right)^2\right) e^{N|x|^2} < 0,$$

where the last inequality follows if N is large enough.

In particular $Lw(x) < 0$, so $w(x)$ is a super-solution.

Now consider

$$h_\epsilon(x) = u(x) - v(x) - \epsilon w(x).$$

Then

$$\begin{array}{ll} Lh_\epsilon(x) > 0 & \text{in } \Omega \\ h_\epsilon(x) \leq -\epsilon w(x) & \text{on } \partial\Omega. \end{array}$$

We may conclude, from Lemma 1, that h_ϵ can not obtain an interior maximum. Thus, for any $\epsilon > 0$,

$$\sup_\Omega \left(u(x) - v(x) - \epsilon w(x)\right) \leq \epsilon \sup_{\partial\Omega}(-w).$$

If we let $\epsilon \to 0$ this implies that

$$\sup_\Omega \left(u(x) - v(x)\right) \leq 0 \Rightarrow u(x) \leq v(x).$$

□

Corollary 2. *Let Ω be a bounded domain and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$\begin{array}{ll} Lu(x) = Lv(x) & \textit{in } \Omega \\ u(x) = v(x) & \textit{on } \partial\Omega. \end{array}$$

Then, if $c(x) \leq 0$, it follows that $u(x) = v(x)$ in Ω.

Proof: By the previous Corollary it follows that $u(x) \leq v(x)$ and $v(x) \leq u(x)$ in Ω. □

Chapter 2

Apriori estimates.

We know that the solutions to

$$\begin{array}{ll} Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array} \tag{2.1}$$

are unique, if they exist. The difficult part is to prove existence. That will take considerable effort. We will start lay the foundations of the existence theory in this chapter. In the next section we will prove sketch a strategy of how to solve the problem. In particular, we will try to motivate the need for apriori estimates. Then we will prove the estimates for the Laplace equation. At the end of the chapter we will prove existence in a very basic case and use that basic case as a springboard for a continued discussion of the strategy.

2.1 Discussion.

We need to find an approach to analyze a very difficult equation. We are in particular interested in showing existence of solutions. One way to approach the problem is to first consider operators L that somehow are close to the Laplace equation - which we can solve. Let us consider

$$L_t u(x) = \Delta u(x) + t\,(L - \Delta)\,u(x),$$

then $L_0 = \Delta$ and $L_1 = L$ so, at least intuitively, $L_t \approx \Delta$ for small t and for $t = 1$ we are back at the general case. If we assume that, for every small t, there exists a solution $u_t(x)$ to the following equation

$$\begin{array}{ll} L_t u_t(x) = f(x) & \text{in } \Omega \\ u_t(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

Then for t small we would expect $u_t(x) \approx u^0(x) + tu^1(x)$ for some functions $u^0(x)$ and $u^1(x)$. What equations would we have to solve to calculate u^0 and

u^1? If we set $t=0$ we get, since $L_0\cdot = \Delta\cdot$,

$$\begin{aligned} L_0 u_0(x) &= \Delta u^0(x) f(x) && \text{in } \Omega \\ u^0(x) &= g(x) && \text{on } \partial\Omega, \end{aligned}$$

which is fine since we know how to solve the Dirichlet problem for the Laplacian. However to calculate $u^1(x)$ we would need to solve, and here I am rather informal,

$$f(x) = L_t u_t(x) \approx \Delta(u^0 + tu^1) + t\,(L-\Delta)\,(u^0 + tu^1) \approx \underbrace{\Delta u^0}_{=f(x)} + t\left(\Delta u^1 + (L-\Delta)\,u^0\right),$$

where we have disregarded terms of order t^2. We see that we need to solve

$$\Delta u^1(x) = (\Delta - L)\,u^0(x), \tag{2.2}$$

Equation (2.2) is in principle fine since we can solve the Dirichlet problem and the right hand side is well defined. But we have only shown that $u^0 \in C^2(\Omega)$ so the right hand side of (2.2) is, as far as we know, only continuous. But we need the right hand side to be C^α to solve (2.2).

In general, we will need to improve our regularity results so that $\Delta u(x) = f(x) \in C^\alpha$ implies that $u \in C^{2,\alpha}$. We will prove this in the next section and also show that these estimates are strong enough to show existence in some simple cases.

2.2 Interior Aproiri Estimates for the Laplacian.

Sinc eour aim in this section is to estimate $\left|D^2u(x) - D^2u(y)\right|$ where $\Delta u(x) = f(x)$ we need to have a better understanding of the Newtonian kernel which we will provide in the next lemma.

Lemma 2. *Let $x, y \in \mathbb{R}^n$, $|x-y| = r$ and*

$$N_{ij}(x) = \frac{\partial^2 N(x)}{\partial x_i \partial x_j}$$

be the second derivatives of the Newtonian kernel. Then,

$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| \leq \frac{C|x-y|}{|x-\xi|^{n+1}}$$

for any $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$.

Proof: Fix a $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$. Then $N(z-\xi) \in C^\infty(\mathbb{R}^n \setminus \{z = \xi\})$. In particular, $N(z-\xi) \in C^\infty(B_{3r/2}(x))$ so we may calculate

$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| = \left|\int_0^1 (x-y)\cdot\nabla N_{ij}(sx + (1-s)y - \xi)ds\right| \leq$$

$$\leq |x-y| \sup_{z\in B_r(x)} |\nabla N_{ij}(z-\xi)|. \tag{2.3}$$

Next we notice that

$$\sup_{z\in B_r(x)} |\nabla N_{ij}(z-\xi)| \leq \sup_{z\in B_{|\xi-x|+r}(0)\setminus B_{|\xi-x|-r}} |\nabla N_{ij}(z)|.$$

But since

$$N(z) = N(x) = \begin{cases} -\frac{1}{2\pi}\ln(|x|) & \text{for } n=2 \\ -\frac{1}{(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{for } n\neq 2, \end{cases}$$

It follows that

$$\sup_{z\in B_{|\xi-x|+r}(0)\setminus B_{|\xi-x|-r}} |\nabla N_{ij}(z)| \leq \frac{C_n}{(|\xi-x|-r)^{n+1}}, \tag{2.4}$$

but if $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$ it clearly follows that

$$|\xi-x|-r \geq \frac{1}{2}|\xi-x|$$

from which we may conclude that

$$\sup_{z\in B_{|\xi-x|+r}(0)\setminus B_{|\xi-x|-r}} |\nabla N_{ij}(z)| \leq \frac{C_n 2^n}{(|\xi-x|)^{n+1}}.$$

Using this last inequality together with (2.4) and (2.3) will result in

$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| \leq \frac{C_n 2^n|x-y|}{(|\xi-x|)^{n+1}}$$

which is the conclusion of the Lemma up to the naming of a constant. □

Theorem 1. *Let $f(x)\in C_c^\alpha(B_{2R}(0))$ for some $0<\alpha<1$ and define*

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi)f(\xi)d\xi$$

then for any $x,y\in B_R(0)$, $x\neq y$, the following inequality holds

$$\frac{\left|\frac{\partial^2 u(x)}{\partial x_i\partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j}\right|}{|x-y|^\alpha} \leq C_{\alpha,n}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{|x-y|^{1-\alpha}\sup_{B_R(0)}|f(x)|}{R}\right). \tag{2.5}$$

In particular, $u\in C^{2,\alpha}(B_R(0))$ and

$$[D^2u]_{C^\alpha(B_R(0)} \leq C_{\alpha,n}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right),$$

where $C_{\alpha,n}$ only depends on the dimension and α.

Proof: We have already shown that $u(x) \in C^2$ and that

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_{2R}(0)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_{2R}(0)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi).$$

We will use this representation to prove (2.5). We set $r = |x-y|$ and calculate

$$\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right| \leq$$

$$\leq \Bigg| \int_{B_{2R}(0)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_{2R}(0)} N_i(x-\xi) \nu_j dA(\xi) -$$

$$- \int_{B_{2R}(0)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi + f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi) \nu_j dA(\xi) \Bigg| \leq$$

$$\leq \left| \int_{B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi \right| + \left| \int_{B_{2r}(x)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi \right| +$$

$$+ \Bigg| \int_{B_{2R}(0) \setminus B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi + \int_{B_{2R}(0) \setminus B_{2r}(x)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi +$$

$$+ f(x) \int_{\partial B_{2R}(0)} N_i(x-\xi) \nu_j dA(\xi) - f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi) \nu_j dA(\xi) \Bigg|$$

$$\leq \left| \int_{B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi \right| + \left| \int_{B_{2r}(y)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi \right| +$$

$$+ \left| \int_{B_{2R}(0) \setminus B_{2r}(x)} \left(N_{ij}(x-\xi) - N_{ij}(y-\xi)\right) \left(f(\xi) - f(x)\right) d\xi \right| + \tag{2.6}$$

$$+ \Bigg| \int_{B_{2R}(0) \setminus B_{2r}(x)} N_{ij}(y-\xi) \left(f(y) - f(x)\right) d\xi +$$

$$- f(x) \int_{\partial B_{2R}(0)} N_i(x-\xi) \nu_j dA(\xi) + f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi) \nu_j dA(\xi) \Bigg|.$$

We will estimate the terms in turn. First we use that $|f(\xi) - f(x)| \leq [f]_{C^\alpha} |\xi - x|^\alpha$ to conclude that

$$\left| \int_{B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi \right| \leq [f]_{C^\alpha} \int_{B_{2r}(x)} |N_{ij}(x-\xi)| \, |\xi - x|^\alpha d\xi \leq$$

$$\leq C [f]_{C^\alpha} \int_{B_{2r}(x)} |\xi - x|^{\alpha - n} d\xi \leq \frac{C[f]_{C^\alpha}}{\alpha} (2r)^\alpha \leq C_\alpha [f]_{C^\alpha} r^\alpha$$

where C_α only depend on α, and n. Similarly we may estimate

$$\left| \int_{B_{2r}(x)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi \right| \leq C_\alpha [f]_{C^\alpha} r^\alpha.$$

Next we use Lemma 2 to estimate

$$\left| \int_{B_{2R}(0)\setminus B_{2r}(x)} \left(N_{ij}(x-\xi) - N_{ij}(y-\xi)\right)\left(f(\xi)-f(x)\right) d\xi \right| \leq$$

$$\leq [f]_{C^\alpha} \int_{B_{2R}(0)\setminus B_{2r}(x)} \frac{C|x-y|}{|x-\xi|^{n-1}} |\xi - x|^\alpha d\xi \leq$$

$$\leq C[f]_{C^\alpha}|x-y| \int_{B_{2R}(0)\setminus B_{2r}(x)} \frac{C}{|x-\xi|^{n+1-\alpha}} d\xi \leq$$

$$\leq C[f]_{C^\alpha} r \left(\frac{1}{(2r)^{1-\alpha}} - \frac{1}{(2R)^{1-\alpha}} \right) \leq C[f]_{C^\alpha} r^\alpha + C[f]_{C^\alpha} \frac{r}{R^{1-\alpha}}.$$

To estimate the final integral in (2.6) we do an integration by parts in the first term and use the triangle inequality as follows

$$\Bigg| \int_{B_{2R}(0)\setminus B_{2r}(x)} N_{ij}(y-\xi)\left(f(y)-f(x)\right) d\xi -$$

$$-f(x)\int_{\partial B_{2R}(0)} N_i(x-\xi)\nu_j dA(\xi) + f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi)\nu_j dA(\xi) \Bigg| =$$

$$\Bigg| -\int_{\partial B_{2R}(0)} N_i(y-\xi)\nu_j \left(f(y)-f(x)\right) d\xi - \int_{\partial B_{2r}(x)} N_i(y-\xi)\nu_j \left(f(y)-f(x)\right) d\xi$$

$$-f(x)\int_{\partial B_{2R}(0)} N_i(x-\xi)\nu_j dA(\xi) + f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi)\nu_j dA(\xi) \Bigg| \leq$$

$$\leq \left| \int_{\partial B_{2r}(x)} N_i(y-\xi)\nu_j \underbrace{\left(f(y)-f(x)\right)}_{\leq [f]_{C^\alpha} r^\alpha} d\xi \right| +$$

$$+|f(x)| \int_{\partial B_{2R}(x)} \underbrace{|N_i(x-\xi) - N_i(y-\xi)|}_{\leq \frac{C|x-y|}{R^n} \text{ on } \partial B_{2R}} dA(\xi) \leq$$

$$\leq C[f]_{C^\alpha} r^\alpha + \frac{C|f(x)|r}{R},$$

notice that we get out an extra minus when we integrte by parts in the first equality since ξ hs a minus in the argument of $N_{ij}(y-\xi)$.

Collecting the terms we arrive at

$$\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right| \leq C\left([f]_{C^\alpha} \left(|x-y|^\alpha + \frac{|x-y|}{R^{1-\alpha}} \right) + \frac{|f(x)||x-y|}{R} \right) \leq$$

$$\leq C\left([f]_{C^\alpha}|x-y|^\alpha + \frac{|x-y| \sup_{B_R(0)} |f(x)|}{R} \right),$$

dividing both sides by $|x-y|^\alpha$ and taking the supremum over all $x, y \in B_R(0)$ gives the desired estimate. □

Observe that the above Theorem only estimates the second derivatives in $B_R(0)$ - that is away from the boundary. For further applications we will however need the estimate close to the boundary.

Proposition 1. *Let Ω be a domain and assume that $u(x)$ is a solution to*

$$\Delta u(x) = f(x) \qquad \text{in } \Omega$$

assume furthermore that $|u| \leq M$ in Ω and that $f \in C^\alpha_{loc}(\Omega)$ and that for any compact set $K \subset \Omega$ the function $f(x)$ satisfies the following estimate

$$\sup_{x \in K} |f(x)| \leq \frac{C_{0,f}}{dist(K, \partial\Omega)^2} \tag{2.7}$$

and

$$|f(x) - f(y)| \leq \frac{C_{\alpha,f}|x-y|^\alpha}{dist(K, \partial\Omega)^{2+\alpha}} \tag{2.8}$$

then there exists a constant $C_{n,\alpha}$ depending only on α and the dimension n such that

$$\sup_{x \in K} |D^2 u(x)| \leq C_{n,\alpha} \frac{C_{0,f} + C_{\alpha,f} + \sup_\Omega |u|}{dist(K, \partial\Omega)^2} \tag{2.9}$$

and

$$\sup_{x,y \in K} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq C_{n,\alpha} \frac{C_{0,f} + C_{\alpha,f} + \sup_\Omega |u|}{dist(K, \partial\Omega)^{2+\alpha}} \tag{2.10}$$

Proof: We will begin by showing (2.9). The proof is not that difficult - even though the result is very technical.

Part 1: *The inequality (2.9) holds.*

We fix a compact set $K \subset \Omega$. Since K is compact and Ω open the distance $\text{dist}(K, \partial\Omega) > 0$, we define $d = \frac{\text{dist}(K,\partial\Omega)}{4} > 0$. Let $x^0 \in K$ be an arbitrary point then $\text{dist} x^0, \partial\Omega \geq 4d$ and the following function

$$v(x) = u(dx + x^0)$$

is well defined in $B_4(0)$. The chain rule implies that

$$\Delta v(x) = d^2 \Delta u(dx + x^0) = d^2 f(dx + x^0) \equiv g(x) \qquad \text{in } B_4(0),$$

where we define $g(x)$ in the last step.

We see that $v(x)$ and $g(x)$ satisfies

$$\sup_{x \in B_4(0)} |v(x)| = \sup_{x \in B_{4d}(x^0)} |u(x)| \leq \sup_{x \in \Omega} |u(x)|,$$

$$\sup_{x \in B_3(0)} |g(x)| = d^2 \sup_{x \in B_{3d}(x^0)} |f(x)| \leq C_{0,f},$$

where we have used (2.7) in the last inequality as well as $\text{dist}(B_{3d}(x^0), \partial\Omega) \geq d$.

Furthermore, we may estimate for $x, y \in B_3(0)$

$$\frac{|g(x) - g(y)|}{|x-y|^\alpha} = d^2 \frac{|f(dx + x^0) - f(dy + x^0)|}{|x-y|^\alpha} =$$

$$\left\{ \begin{array}{l} \text{substitute} \\ \tilde{x} = dx + x^0,\ \tilde{y} = dy + x^0 \end{array} \right\} = d^{2+\alpha} \frac{|f(\tilde{x}) - f(\tilde{y})|}{|\tilde{x} - \tilde{y}|^\alpha} \leq$$

$$\leq d^{2+\alpha} C_{\alpha,f} \underbrace{\text{dist}(B_{3d}(x^0), \partial\Omega)^{-(2+\alpha)}}_{\leq d^{-(2+\alpha)}} \leq C_{\alpha,f},$$

where we again used that $\text{dist}(B_{3d}(x^0), \partial\Omega) \geq d$ in the last inequality.

We have thus shown that $v(x)$ solves the following Dirichlet problem

$$\begin{array}{ll} \Delta v(x) = g(x) & \text{in } B_2(0) \\ v(x) = u(dx + x^0) & \text{on } \partial B_2(0), \end{array}$$

where $v(x)$ and $g(x)$ are is bounded by $\sup_\Omega |u|$ and $C_{0,f}$ respectively and $[g]_{C^\alpha(B_3(0))} \leq C_{\alpha,f}$.

Next we let $\varphi(x) \in C_c^\infty(B_3(0))$ be such that $\varphi = 1$ in $B_2(0)$ and $|\nabla\varphi| \leq 2$.[1] We also define

$$w(x) = \int_{\mathbb{R}^n} N(x-y) g(y) \varphi(y) dy,$$

where $N(x-y)$ is the Newtonian kernel, notice that the integral is well defined since $g(y)\varphi(y) = 0$ outside of $B_3(0)$. Clearly,

$$\sup_{B_2(0)} |w(x)| \leq \sup_{B_3(0)} |g(x)| \int_{B_2(0)} N(x-y) dy \leq C_n \sup_{B_3(0)} |g(x)| = C_n C_{0,f},$$

where the constant only depend on the dimension. Furthermore, by the estimates in Theorem 1 in the first set of notes[2] we know that

$$\left| \frac{\partial^2 w(0)}{\partial x_i \partial x_j} \right| =$$

$$= \left| \int_{B_3(0)} \frac{\partial^2 N(y)}{\partial x_i \partial x_j} \left(g(y)\varphi(y) - g(0) \right) dy - g(0) \int_{\partial B_3(x)} \frac{\partial N(y)}{\partial x_i} \nu_j(\xi) dA(y) \right|$$

$$\leq \left| \int_{B_3(0)} \frac{\partial^2 N(y)}{\partial x_i \partial x_j} \left(g(y)\varphi(y) - g(0) \right) dy \right| +$$

[1] That this is possible is easy to see geometrically, or one could define $\varphi(x) = \int_{B_{5/2}(0)} \phi_{1/4}(x-y) dy$ where $\phi_{1/4}$ is the standard mollifier.

[2] See step 3 of that proof.

$$+|g(0)|\underbrace{\left|\int_{\partial B_3(x)}\frac{\partial N(y)}{\partial x_i}\nu_j(\xi)dA(y)\right|}_{\substack{\leq 1 \text{ by direct}\\ \text{calculation}}}$$

$$\leq (C_{\alpha,f}+C_{0,f})\int_{B_3(0)}\frac{1}{|y|^{n-\alpha}}dy+C_{0,f}\leq C_n\left(C_{\alpha,f}+C_{0,f}\right)$$

We can conclude that

$$\begin{array}{ll}\Delta w(x)=g(x) & \text{in } B_2(0)\\ |D^2w(0)|\leq C_n\left(C_{\alpha,f}+C_{0,f}\right) & \text{and}\\ \sup_{B_2(0)}|w(x)|\leq C_nC_{0,f}. & \end{array}$$

This in turn implies that $h(x)=v(x)-w(x)$ satisfies

$$\Delta h(x)=0 \quad \text{in } B_2(0)$$

$$\sup_{B_2(0)}|h(x)|\leq \sup_{B_2(0)}|v(x)|+\sup_{B_2(0)}|w(x)|\leq \sup_{\Omega}|u(x)|+C_nC_{0,f}.$$

In particular, we can conclude from our interior regularity for harmonic functions that

$$|D^2h(0)|\leq \frac{n^3 2^{2n+4}}{\omega_n 2^{n+2}}\|h\|_{L^1(B_2(0))}\leq C_n\left(\sup_{\Omega}|u(x)|+C_{0,f}\right).$$

We may conclude that

$$\left|D^2v(0)\right|=\left|D^2\left(v(0)-w(0)+w(0)\right)\right|=$$

$$=\left|D^2\left(h(0)+w(0)\right)\right|\leq C_{n,\alpha}\left(\sup_{\Omega}|u(x)|+C_{0,f}+C_{\alpha,f}\right).$$

But

$$D^2v(0)=d^2D^2u(x^0)$$

which implies

$$\left|D^2u(x^0)\right|\leq\frac{C_{n,\alpha}\left(\sup_\Omega|u(x)|+C_{0,f}+C_{\alpha,f}\right)}{d^2}=$$

$$=\frac{16C_{n,\alpha}\left(\sup_\Omega|u(x)|+C_{0,f}+C_{\alpha,f}\right)}{\operatorname{dist}(K,\partial\Omega)^2}.$$

This proves part 1.

Part 2: *The inequality (2.10) holds.*

We use the same set-up as in part 1 and let $K\subset\Omega$ be a compact set and $x^0,y^0\in K$ be arbitrary points. First we notice that if $|x^0-y^0|\geq d$ then the estimate immediately follows, indeed:

$$\frac{|D^2u(x^0)-D^2u(y^0)|}{|x^0-y^0|^\alpha}\leq\frac{|D^2u(x^0)|+|D^2u(y^0)|}{|x^0-y^0|^\alpha}\leq$$

$$\leq \frac{2^{1+2\alpha} C_{n,\alpha} \left(\sup_\Omega |u(x)| + C_{0,f} + C_{\alpha,f}\right)}{\text{dist}(K, \partial\Omega)^2 |x^0 - y^0|^\alpha} \leq \frac{2C_{n,\alpha} \left(\sup_\Omega |u(x)| + C_{0,f} + C_{\alpha,f}\right)}{\text{dist}(K, \partial\Omega)^{2+\alpha}},$$

which is the desired estimate. Therefore we may assume, without loss of generality, that $|x^0 - y^0| < d$.

We define $v(x)$ as in part 1 of this proof. Then, with $z^0 = \frac{y^0 - x^0}{d} \in B_1(0)$

$$\frac{\left|D^2 v(0) - D^2 v(z^0)\right|}{|0 - z^0|^\alpha} = d^{2+\alpha} \frac{|D^2 u(x^0) - D^2 u(y^0)|}{|x^0 - y^0|^\alpha}.$$

Therefore it is enough to show that

$$\frac{\left|D^2 v(0) - D^2 v(z^0)\right|}{|z^0|^\alpha} \leq C_{n,\alpha} \left(C_{0,f} + C_{\alpha,f} + \sup_\Omega |u|\right).$$

If we define $w(x)$ and $h(x)$ as in Part 1 of this proof then it follows from Theorem 1, in particular from (1), that

$$\frac{\left|\frac{\partial^2 w(0)}{\partial x_i \partial x_j} - \frac{\partial^2 w(z^0)}{\partial x_i \partial x_j}\right|}{|z^0|^\alpha} \leq C_{\alpha,n} \left(C_{\alpha,f} + C_{0,f}\right),$$

where we have used that $|z^0| < 1$, that $[g]_{C^\alpha(B_2(0))} \leq C_{\alpha,f}$ and $\sup_{B_2(0)} |g| \leq C_{0,f}$.

Next we estimate

$$\frac{\left|\frac{\partial^2 h(0)}{\partial x_i \partial x_j} - \frac{\partial^2 h(z^0)}{\partial x_i \partial x_j}\right|}{|z^0|^\alpha} \leq \frac{\sup_{x \in B_1(0)} |D^3 h(x)||z_0|}{|z^0|^\alpha} \leq \sup_{x \in B_1(0)} |D^3 h(x)|$$

where we used the mean value theorem for the derivative to conclude that $\frac{\partial^2 h(0)}{\partial x_i \partial x_j} - \frac{\partial^2 h(z^0)}{\partial x_i \partial x_j} = z^0 \cdot \nabla \frac{\partial^2 h(\xi)}{\partial x_i \partial x_j}$ for some ξ on the line from the origin to z^0. But since $h(x)$ is harmonic in $B_2(0)$ it follows that

$$\sup_{B_1(0)} |D^3 h(x)| \leq C_n \|h\|_{L^1(B_2(0))} \leq C_n \sup_{B_2(0)} |h(x)| \leq C_n \left(\sup_\Omega |u(x)| + C_{0,f}\right).$$

We can thus conclude that

$$\frac{\left|D^2 v(0) - D^2 v(z^0)\right|}{|z^0|^\alpha} \leq \frac{\left|D^2 w(0) - D^2 w(z^0)\right|}{|z^0|^\alpha} +$$

$$+ \frac{\left|D^2 h(0) - D^2 h(z^0)\right|}{|z^0|^\alpha} \leq C_{n,\alpha} \left(C_{0,f} + C_{\alpha,f} + \sup_\Omega |u|\right).$$

This finishes the proof. □

2.2.1 An application.

Before we consider the general case of an elliptic PDE we will consider a simpler perturbation result with a PDE that is some sense is close to the Laplace equation. We will improve on the following result significantly later in the course.

In this section we will assume that

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \qquad \text{in } B_1(0), \tag{2.11}$$

where a_{ij} satisfies the ellipticity condition and the following conditions

$$\|a_{ij}\|_{C^\alpha(B_1(0))} \le \epsilon \qquad \text{for } i,j = 1,2,...,n \text{ and } i \ne j \tag{2.12}$$

and

$$\|a_{ii} - 1\|_{C^\alpha(B_1(0))} \le \epsilon \qquad \text{for } i = 1,2,...,n. \tag{2.13}$$

The conditions (2.12) and (2.13) means that the partial differential operator is close to Laplace in some sense. In particular, if $\epsilon = 0$ then $L = \Delta$.

Lemma 3. *Let $f(x) \in C^\alpha(B_1(0))$ and $g(x) \in C(\partial B_1(0))$. Assume furthermore that L is as in (2.11) and that $a_{ij}(x)$ satisfies (2.12)-(2.13). Assume furthermore that there exists a $\delta > 0$ such that*

$$a_{ij}(x) = 0 \qquad \text{if } x \in B_1(0) \setminus B_{1-\delta}(0) \text{ and } i \ne j,$$

and

$$a_{ii}(x) = 1 \qquad \text{if } x \in B_1(0) \setminus B_{1-\delta}(0),$$

that is $L\cdot = \Delta\cdot$ in $B_1(0) \setminus B_{1-\delta}(0)$.

Then there exists an $\epsilon_\delta > 0$ (depending on $\delta > 0$ as well as f, g, a_{ij} and Ω) such that if $\epsilon < \epsilon_\delta$ then there exists a unique solution to

$$\begin{aligned} Lu(x) &= f(x) && \text{in } \Omega \\ u(x) &= g(x) && \text{on } \partial\Omega. \end{aligned}$$

Proof: Even though we have made many preparations the proof is quite complicated. We will prove the Lemma by constructing a convergent sequence of approximating solutions starting with the Dirichlet problem.

Observe that we can find a solution, $u^0(x)$, to

$$\begin{aligned} \Delta u^0(x) &= f(x) && \text{in } \Omega \\ u^0(x) &= g(x) && \text{on } \partial\Omega. \end{aligned}$$

We will inductively define $u^k(x)$, for $k = 1,2,...$, as the solution to

$$\begin{aligned} \Delta u^k(x) &= \Delta u^{k-1} - Lu^{k-1}(x) + f(x) && \text{in } \Omega \\ u^k(x) &= g(x) && \text{on } \partial\Omega. \end{aligned}$$

Since we are going to work with the differences $u - u^{k-1}$ for most of the proof we define $w^k(x) = u^k(x) - u^{k-1}(x)$ for $k \ge 1$ and $w^0(x) = u^0(x)$.

Then

$$\Delta w^k(x) = \Delta u^{k-1}(x) - Lu^{k-1}(x) - \Delta u^{k-2}(x) + Lu^{k-2}(x) = \tag{2.14}$$
$$= (\Delta - L)\left(w^{k-1}(x)\right).$$

But on each compact set $K \subset \Omega$ we have, by Proposition 1 in particular (2.9), that

$$\sup_{x\in K} |D^2 w^k(x)| \leq \tag{2.15}$$
$$\leq C_{n,\alpha} \frac{d^2 \sup_{x\in K} |(\Delta - L)w^{k-1}| + d^{2+\alpha}\left[(\Delta - L)(w^{k-1})\right]_{C^\alpha(K)} + \sup_\Omega |w^k|}{d^2}$$

where $d = \operatorname{dist}(K, \partial\Omega)$ and similarly

$$\left[D^2 w^k(x)(x))\right]_{C^\alpha(K)} \leq \tag{2.16}$$
$$\leq C_{n,\alpha} \frac{d^2 \sup_{x\in K} |(\Delta - L)w^{k-1}| + d^{2+\alpha}[(\Delta - L)w^{k-1}]_{C^\alpha(K)} + \sup_{B_1(0)} |w^k|}{d^{2+\alpha}}.$$

But clearly, with the notation $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$,

$$\sup_{x\in K} |(\Delta - L)w^{k-1}| \leq$$
$$\underbrace{\sup_{i,j=1,\dots,n}\left(\sup_{x\in K} |a_{ij}(x) - \delta_{ij}|\right)}_{\leq \epsilon} \sup_{x\in K} |D^2 w^{k-1}(x)| \leq$$
$$\leq \epsilon \sup_{x\in K} |w^{k-1}(x)|$$

and similarly[3]

$$\left[(\Delta - L)(w^{k-1})\right]_{C^\alpha(K)} \leq \epsilon\left[D^2 w^{k-1}\right]_{C^\alpha(K)}.$$

In conclusion we get form (2.15) and (2.16) that

$$\sup_{x\in K} |D^2 w^k(x)| \leq \tag{2.17}$$
$$\leq C_{n,\alpha}\epsilon\left(\sup_{x\in K} |D^2 w^{k-1}(x)| + d^\alpha\left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) + \frac{\sup_{B_1(0)} |w^k|}{d^2}$$

and

$$\left[|D^2 w^k(x)\right]_{C^\alpha(K)} \leq \tag{2.18}$$
$$\leq \frac{C_{n,\alpha}\epsilon}{d^\alpha}\left(\sup_{x\in K} |D^2 w^{k-1}(x)| + d^\alpha\left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) + \frac{\sup_{B_1(0)} |w^k|}{d^{2+\alpha}}$$

[3]Here we are cheating a little. To be exact, we are skipping some details. The assertion is justified but it uses some results that we will cover later.

We need to estimate $\sup_\Omega |w^k|$. Notice that $w^k(x) = u^k - u^{k-1} = 0$ on $\partial B_1(0)$ and from (2.14) we get that

$$\left|\Delta w^k\right| = \left|(\Delta - L)\, w^{k-1}(x)\right| \leq$$
$$\leq \epsilon \sup_{x \in B_{1-\delta}(0)} \left|D^2 w^{k-1}(x)\right|.$$

Therefore, by the comparison principle we can deduce that

$$-B(x) \leq w^k(x) \leq B(x) \tag{2.19}$$

where

$$B(x) = \frac{\epsilon \sup_{x \in B_{1-\delta}(0)} \left|D^2 w^{k-1}(x)\right|}{2n}(1 - |x|^2).$$

In particular $B(x) = w^k(x) = 0$ on $\partial B_1(0)$ and $-\Delta B(x) \geq \Delta w^k \geq \Delta B(x)$.

We may thus estimate

$$\sup_{B_1(0)} |w^k(x)| \leq \epsilon \sup_{x \in B_{1-\delta}(0)} \left|D^2 w^{k-1}(x)\right|. \tag{2.20}$$

Using (2.20) in (2.17) and (2.18) we can conclude that

$$\sup_{x \in K} |D^2 w^k(x)| + d^\alpha \left[D^2 w^k(x)\right]_{C^\alpha(K)} \leq$$
$$\leq C\epsilon \left(\frac{1 + d^2}{d^2} \sup_{x \in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) \leq \tag{2.21}$$
$$\leq \frac{1}{2}\left(\sup_{x \in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right)$$

where the last inequality follows if ϵ is small enough, say $\epsilon \leq \frac{c\delta^2}{2}$ for some small c, and $d \geq \delta$.

Equation (2.21) is the heart of the proof since it shows that

$$\sup_{x \in K} |D^2 w^k(x)| + d^\alpha \left[D^2 w^k(x)\right]_{C^\alpha(K)} \leq$$
$$\leq \frac{1}{2}\left(\sup_{x \in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) \leq$$
$$\leq \frac{1}{2^2}\left(\sup_{x \in K} |D^2 w^{k-2}(x)| + d^\alpha \left[D^2 w^{k-2}\right]_{C^\alpha(K)}\right) \leq$$
$$\leq \cdots \leq \frac{1}{2^k}\left(\sup_{x \in K} |D^2 w^0(x)| + d^\alpha \left[D^2 w^0\right]_{C^\alpha(K)}\right).$$

That is $D^2 u^k(x)$ forms a Cauchy sequence in $C^\alpha(K)$ for any $K \in B_{1-\delta}(0)$. So, by the Arzela-Ascoli Theorem, $u^k \to u$ uniformly on $B_{1-\delta}(0)$. And on

$B_1(0) \setminus B_{1-\delta}(0)$ it directly follows that u^k converges, at least for a subsequence, since $\Delta u^k(x) = f(x)$ in $B_1(0) \setminus B_{1-\delta}(0)$.

So $u^k \to u$ in $C^{2,\alpha}_{\text{loc}}(B_1(0))$. By the definition of u^k it als follows that

$$\Delta u^k = (\Delta - L)u^{k-1}(x) + f(x) \Rightarrow Lu^{k-1} = \underbrace{\Delta(u^{k-1}(x) - u^k(x))}_{\to 0 \text{ in } C^\alpha_{\text{loc}}} + f(x).$$

It follows that

$$Lu(x) = f(x) \qquad \text{in } B_1(0).$$

Using (2.19) together with

$$|w^k(x)| \le B(x) \le \frac{C(1-|x|^2)}{2^k}$$

we can conclude that

$$|u^k(x) - u^0(x)| \le \sum_{i=1}^k |u^k - u^{k-1}| \le \sum_{i=1}^k |w^k(x)| \le C(1-|x|^2) \sum_{i=1}^k \frac{1}{2^i} \le C(1-|x|^2)$$

which implies that for every $k = 1, 2, 3, ...$

$$u^0(x) - C(1-|x|^2) \le u^k(x) \le u^0(x) + C(1-|x|^2).$$

Since $u^0 \in C(\overline{\Omega})$ we can conclude that $u(x) \in C(\overline{\Omega})$.

Uniqueness of $u(x)$ follows by the maximum principle, in particular Corollary 2. □

Remarks: There are several things to say about this Lemma.

1. First of all, the result is very unsatisfactory in several respects. The most obvious is that we assume that $a_{ij}(x) = \delta_{ij}$ in $B_1(0) \setminus B_{1-\delta}(0)$. But this assumption is necessary for us to estimate (2.21).

   That we need this assumption is because we our estimates of $D^2u(x)$ breaks down when x is close to the boundary. In particular the presence of the inverse of the distance to the boundary in the statement of Proposition 1.

   Therefore we need to develop a theory that better estimates the solutions close to the boundary, estimates without the $\text{dist}(K, \partial\Omega)^{-1}$ terms.

2. We need to develop some better terminology in order not to get lost in the technicalities. In particular we need to develop the language of Banach spaces as well as some functional analysis.

3. It is also rather unsatisfactory that the proof only works for small ϵ. The theory only works for small ϵ because, and this is very important, we do not have a regularity theory for the general equation $Lu(x) = f(x)$. If we had such a theory we could, using the terminology of the beginning of this chapter, apply the same proof in Lemma 3 to find a solution to $L_{t+\epsilon}u(x) = f(x)$ if we could solve a solution to $L_t u(x) = f(x)$ - and then for $L_{t+2\epsilon}u(x) = f(x)$ etc.

In the next chapter we will continue to develop the regularity theory for elliptic PDE. Then we will see that the interior regularity theory actually implies boundary regularity. Once we have the regularity theory in place we will be able to show existence for the general equation $Lu(x) = f(x)$ - under some assumptions on L and on the boundary of Ω.