Selected Topics in PDE part 6.

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Chapter 1

Apriori interior estimates for constant coefficient PDE.

In the last chapter we saw that we can estimate $[D^2u]_{C^\alpha}$ for the solution to $\Delta u(x) = f(x)$ in terms of f and $\sup|u|$. And very importantly, we also saw that such estimates leads to existence of solutions for PDE with coefficients that are close, in $C^\alpha-$norm, to that are close to the coefficients of Δ (that is $a_{ij}(x) \approx \delta_{ij}$). We will use this knowledge to construct solutions to general variable coefficients PDE.

In particular, if we consider a general linear PDE with variable coefficients:

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \quad \text{in } \Omega, \tag{1.1}$$

where $a_{ij}(x), b_i(x)$ and $c(x) \in C^\alpha$. Then if we consider a small enough ball $B_r(x^0) \subset \Omega$ then

$$a_{ij}(x) \approx a_{ij}(x^0),\ b_i(x) \approx b_i(x^0) \text{ and } c(x) \approx c(x^0) \qquad \text{in } B_r(x^0).$$

This means that in the small ball $B_r(x^0)$ we will have that $L\cdot$ is close to a PDE with constant coefficients:

$$Lu(x) \approx \sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x^0)\frac{\partial u(x)}{\partial x_i} + c(x^0)u(x) \approx f(x) \quad \text{in } B_r(x^0). \tag{1.2}$$

One usually say that a PDE like (1.2) has *frozen coefficients* and the method we will use is often called *freezing of the coefficients*.

Thus if we understand constant coefficient PDE better then we should be able to better understand a variable coefficient equation. The method is quite subtle, and it is not at all clear at this point that freezing of the coefficients will yield any useful results. However, in this chapter we will prove a simple regularity result for constant coefficient equations. In the next chapter we will show that we can actually freeze the coefficients to get a good regularity theory for variable coefficient equations.

Before reading the rest of this chapter it is advisable to read the appendixes on Banach spaces and interpolation inequalities.

Proposition 1. *Assume that Ω is a bounded domain and that $u(x) \in C^2(\Omega)$ solves the following constant coefficient PDE*

$$\sum_{ij=1}^{n} a_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) \qquad \text{in } \Omega$$

where a_{ij} are constants satisfying the following ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \tag{1.3}$$

for some constants $\Lambda, \lambda > 0$ and all $\xi \in \mathbb{R}^n$.

Then, for any $0 < \alpha < 1$ there exists a constant $C = C(\lambda, \Lambda, n, \alpha)$ such that

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \leq C \left(\|u\|_{C(\Omega)} + \|f\|_{C^{\alpha}_{int,(2)}(\Omega)} \right).$$

Proof: The proof is very simple. We will show that a change of variables transforms $u(x)$ into a harmonic function $v(x)$ and the estimates for $u(x)$ follows from the corresponding estimates for harmonic functions. We will do the proof in several steps - some of them we will only sketch.

Step 1: *We may change variables to transform $u(x)$ into a harmonic function.*

Since the matrix $A = [a_{ij}]$ is symmetric we may write it as

$$A = O^T D O,$$

where O is an orthogonal matrix (with rows consisting of the eigenvectors of A) and D is the diagonal matrix with the eigenvalues of A along the diagonal. Using that A is elliptic, (1.3), we know that the eigenvalues of A are bounded from above and below by Λ and $\lambda > 0$ and we may thus take the square root of D. Now define $P = \sqrt{D} O$, then it follows that $A = P^T P$. Expressed in terms of components:

$$a_{jk} = \sum_{i=1}^{n} p_{ij} p_{ik}.$$

So if we define

$$v(x) = u(Px)$$

then

$$\sum_{i=1}^{n} \frac{\partial^2 v(x)}{\partial x_i^2} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} p_{ij} \frac{\partial u(Px)}{\partial x_j} \right) =$$

$$= \sum_{i,j,k=1}^{n} p_{ij} p_{ik} \frac{\partial^2 u(Px)}{\partial x_j \partial x_k} = \sum_{j,k=1}^{n} \underbrace{\left(\sum_{i=1}^{n} p_{ij} P_{ik} \right)}_{=a_{jk}} \frac{\partial^2 u(Px)}{\partial x_j \partial x_k} =$$

$$= \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u(Px)}{\partial x_j \partial x_k} = f(Px).$$

Thus it follows that $\Delta v(x) = f(Px)$.

It follows from Proposition 1 (Part 5 of these notes, also reformulated in Proposition 2 in the appendix) that

$$\|v\|_{C^{2,\alpha}_{\text{int}}(\Omega)} \leq C \left(\|u\|_{C(\Omega)} + \|f(P\cdot)\|^{(2)}_{C^{\alpha}_{\text{int},(2)}(\Omega)} \right). \tag{1.4}$$

Step 2: *Bound of $|\nabla u(x)|$ on compact sets.*

Since P is an orthogonal matrix times a diagonal matrix with diagonal elements in $[\sqrt{\lambda}, \sqrt{\Lambda}]$ it follows that P is invertible. We may therefore write

$$u(x) = v(P^{-1}x).$$

In particular,

$$\nabla u(x) = P^{-1} \cdot \nabla v(P^{-1}x),$$

But since all eigenvalues of P^{-1} lay in the interval $[\Lambda^{-1/2}, \lambda^{-1/2}]$ it follows that

$$|\nabla u(x)| \leq \frac{1}{\sqrt{\lambda}} |\nabla v(P^{-1/2}x)|. \tag{1.5}$$

Now for any compact set $K \subset \Omega$ we have that

$$P(K) = \{Px;\ x \in K\} \subset P(\Omega) = \{Px;\ x \in \Omega\},$$

and if

$$\text{dist}(K, \partial\Omega) = d \quad \text{ then } \text{dist}(P(K), \partial P(\Omega)) \geq \sqrt{\lambda} d. \tag{1.6}$$

In particular for any $x \in K \subset \Omega$ it follows (1.4), (1.5) and (1.6) that

$$|\nabla u(x)| \leq \frac{C}{\lambda} \frac{\left(\|u\|_{C(\Omega)} + \|f(P\cdot)\|^{(2)}_{C^{\alpha}_{\text{int},(2)}(\Omega)} \right)}{\text{dist}(K, \partial\Omega)}. \tag{1.7}$$

Step 3: *Estimates for $D^2 u(x)$ and $[D^2 u]_{C^\alpha(K)}$.*

This works exactly the same as in step 1. That is we may write $D^2 u$ and $[D^2 u]_{C^\alpha}$ in terms of v and use (1.4). □

Chapter 2

Apriori interior estimates for PDE with variable coefficients.

We are now ready to prove interior apriori estimates[1] for equations with variable coefficients. We will prove the following estimate

$$\|u\|_{C^{2,\alpha}_{\text{int}}(\Omega)} \le C\left(\|f\|_{C^{\alpha}_{\text{int},(2)}} + \|u\|_{C(\Omega)}\right)$$

where $C = C(n, \alpha, \Omega, \lambda, \Lambda)$ and on the coefficients in the equation:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \tag{2.1}$$

We have already seen all the ideas that we are going to use. Our method of proof will be to freeze the coefficients. In particular, if the coefficients of the equation are close to constant, say that $|a_{ij}(x) - a_{ij}(x^0)| \le \epsilon$ for some small $\epsilon > 0$ in a ball $B_r(x^0)$ then we may write equation (2.1) as

$$\sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) - \sum_{i,j=1}^{n} \underbrace{\left(a_{ij}(x) - a_{ij}(x^0)\right)}_{\le \epsilon}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \tag{2.2}$$

$$-\sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} - c(x)u(x) \quad \text{ in } B_r(x^0)$$

[1]An apriori estimate is an estimate for an equation that is made before we know that solutions exist. Typically one assumes that there exist a solution in some Banach space, say $C^{2,\alpha}_{\text{int}}(\Omega)$, and then proves that there is a bound on the norm of that space that does not depend on the solution.

We can view this as a constant coefficient equation (with right hand side depending on u) and apply Proposition 1 and derive that

$$\|u\|_{C^{2,\alpha}_{\text{int}}(B_r(x^0))} \leq C\left(\|u\|_{C(B_r(x^0))} + \|F\|_{C^{\alpha}_{\text{int},(2)}(B_r(x^0))}\right),$$

where $F(x)$ is the right hand side in (2.2). Now $\|F\|_{C^{\alpha}_{\text{int},(2)}(B_r(x^0))}$ will depend on u. But since we multiply the second derivatives of the $u-$term by something of order ϵ in (2.2) the dependence will not be significant if ϵ is small enough.

Therefore we can estimate the $C^{2,\alpha}_{\text{int}}(B_r(x^0))$ (or even the norm in Ω) if $a_{ij}(x) \approx a_{ij}(x^0)$. But, and here is the second main idea[2], if the coefficients are continuous then $|a_{ij}(x) - a_{ij}(x^0)| \leq \epsilon$ in $B_r(x^0)$ for any x^0 if $r > 0$ is small enough. And since we can cover any compact set $K \subset \Omega$ by finitely many balls $B_r(x)$ it is enough to do prove the regularity in a small ball.

We are now ready to state and prove the Theorem.

Theorem 1. *Let $u \in C^{2,\alpha}_{int}(\Omega)$, where Ω is a bounded domain and $\alpha \in (0,1)$, be a solution to*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \qquad \text{in } \Omega.$$

Assume furthermore that $a_{ij}(x), f(x) \in C^{\alpha}(\Omega)$ and that $a_{ij}(x)$ satisfy the ellipticity condition $\lambda|\xi|^2 \leq \sum_{ij} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$. Then there exists a constant $C = C(n,\alpha,\Omega,\lambda,\Lambda,a_{ij})$ such that

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \leq C\left(\|f\|_{C^{\alpha}_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)}\right).$$

Proof: Let $K \subset \Omega$ be a compact set. We need to show that

$$\sum_{j=0}^{2}\left(\text{dist}\,(K,\partial\Omega)^j \sup_{x\in K}|D^j u(x)|\right) + \text{dist}\,(K,\partial\Omega)^{2+\alpha} \sup_{x,y\in K}\frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\alpha}} \leq$$

$$\leq C\left(\|f\|_{C^{\alpha}_{\text{int},(2)}(\Omega)} + \|u\|_{C(\Omega)}\right).$$

But by the interpolation inequality (Proposition 4 in the appendix.) it is enough to show that

$$\text{dist}\,(K,\partial\Omega)^{2+\alpha} \sup_{x,y\in K}\frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\alpha}} \leq C\left(\|f\|_{C^{\alpha}_{\text{int},(2)}} + \|u\|_{C(\Omega)}\right). \quad (2.3)$$

We will prove the Theorem in three steps. First we will cover K by balls $B_{\delta_K}(x^k)$ in a very specific way, then we will prove (2.3) for an ball $B_{\delta}(x^k)$. In the final step we will show that it is enough to prove the Theorem for the balls $B_{\delta_K}(x^k)$ in order to prove the Theorem.

Step 1: *Let $K \subset \Omega$ be a compact set and $\epsilon > 0$ be a fixed constant (to be determined later) depending only on the coefficients of L. Then we may cover K by a balls $B_{\delta_K}(x^k)$. Where the balls $B_{\delta_K}(x^k)$ may be chosen to satisfy*

[2]Freezing of the coefficients was the first

1. $B_{4\delta_K}(x^k) \subset \Omega$,
2. $|a_{ij}(x) - a_{ij}(x^k)| < \epsilon$ *in* $B_{\delta_K}(x^k)$,
3. $\delta_K \geq \frac{\mathrm{dist}(K,\partial\Omega)}{4}$ if $\mathrm{dist}(K, \partial\Omega)$ is small enough.

Since $\|a_{ij}\|_{C^\alpha(\Omega)} < \infty$ there is a $\mu_\epsilon > 0$ such that for any $x \in \Omega$ we have $|a_{ij}(x) - a_{ij}(y)| < \epsilon$ for every $y \in B_{\mu_\epsilon}(x)$. Now let us denote

$$d_K = \frac{\mathrm{dist}(K, \partial\Omega)}{4}$$

and $\delta = \min(d_K, \mu_\epsilon)$. Then obviously $K \subset \cup_{x \in K} B_\delta(x)$. Since K is compact we can find a finite sub-cover $B_\delta(x^k)$ as described in step 1.

Step 2: *The following estimate holds*

$$\delta_K^2 \|D^2 u\|_{C^\alpha_{\mathrm{int}}(B_{2\delta_K}(x^k))} \leq C_L \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} \right),$$

where C_L *depend on the coefficients* a_{ij}, b_i *and* c *through their* $C^\alpha(\Omega)-$*norm and the ellipticity constants* λ, Λ *and also on the dimension* n.

Here we use the freezing of the coefficients argument and write, in the ball $B_{2\delta_K}(x^k)$

$$\sum_{i,j=1}^n a_{ij}(x^0) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) - \sum_{i,j=1}^n \left(a_{ij}(x) - a_{ij}(x^0) \right) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} -$$

$$- \sum_{i=1}^n b_i(x) \frac{\partial u(x)}{\partial x_i} - c(x) u(x) = F(x),$$

where $F(x)$ is defined by the last inequality.

Viewing this a s a constant coefficient PDE we may use Proposition 1 to deduce that

$$\delta_K^2 \|D^2 u\|_{C^\alpha_{\mathrm{int}}(B_{2\delta_K}(x^k))} \leq C \left(\|u\|_{C(\Omega)} + \|F\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} \right) \leq$$

$$\leq C \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} + \left\| \sum_{i,j=1}^n \left(a_{ij}(x) - a_{ij}(x^0) \right) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} \right) +$$

$$+ C \left(\left\| \sum_{i=1}^n b_i(x) \frac{\partial u(x)}{\partial x_i} \right\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c(x) u(x)\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} \right) \leq$$

$$\leq C \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} \right) +$$

$$+ \sum_{i,j=1}^{n} \underbrace{\left\| a_{ij}(x) - a_{ij}(x^0) \right\|_{C(B_{2\delta_K})}}_{<\epsilon} \left\| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right\|_{C_{\mathrm{int},(2)}(B_{2\delta_K})} + \tag{2.4}$$

$$+ \sum_{i,j=1}^{n} \underbrace{\left\| a_{ij}(x) - a_{ij}(x^0) \right\|_{C(B_{2\delta_K})}}_{<\epsilon} \left[\frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right]_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} +$$

$$+ \sum_{i,j=1}^{n} \left[a_{ij}(x) - a_{ij}(x^0) \right]_{C^{\alpha}(B_{2\delta_K})} \left\| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right\|_{C_{\mathrm{int},(2)}(B_{2\delta_K})} +$$

$$+C \left(\sum_{i=1}^{n} \|b_i(x)\|_{C^{\alpha}(\Omega)} \left\| \sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i} \right\|_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c\|_{C(\Omega)} \|u(x)\|_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} \right) \leq$$

$$< C \left(\|u\|_{C(\Omega)} + \|f\|_{C^{\alpha}_{\mathrm{int},(2)}(\Omega)} \right) +$$

$$+C \left(\sum_{i=1}^{n} \|b_i(x)\|_{C(\Omega)} \left\| \sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i} \right\|_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c\|_{C(\Omega)} \|u(x)\|_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} \right) +$$

$$+C_L \delta_K^2 \left\| D^2 u \right\|_{C_{\mathrm{int}}(B_{2\delta_K})} + \underbrace{C\epsilon \delta_K^2 \left[D^2 u \right]_{C_{\mathrm{int}}(B_{2\delta_K})}}_{\leq \frac{\delta_K^2}{4} [D^2 u]_{C_{\mathrm{int}}(B_{2\delta_K})}},$$

where the constant C_L depend on the coefficients a_{ij}, b_i and c through their $C^{\alpha}(\Omega)-$norm and the ellipticity constants λ, Λ and also on the dimension n. We have also used that $[\cdot]_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} \leq C\delta_K^2 [\cdot]_{C^{\alpha}_{\mathrm{int}}(B_{2\delta_K})}$ and the final "underbrace" holds if ϵ is small enough.

Using Propoisition 4 in the appendix we can deduce that

$$C_L \delta_K^2 \left\| D^2 u \right\|_{C_{\mathrm{int}}(B_{2\delta_K})} \leq$$

$$\leq C_L C_{\epsilon} \delta_K^2 \| D^2 u \|_{C_{\mathrm{int}}(B_{2\delta_K})} + \epsilon C_L \delta_K^2 \left[D^2 u \right]_{C^{\alpha}_{\mathrm{int}}(B_{2\delta_K})} \leq \tag{2.5}$$

$$\leq C_L C_{\epsilon} \delta_K^2 \| D^2 u \|_{C_{\mathrm{int}}(B_{2\delta_K})} + \frac{\delta_K^2}{4} \left[D^2 u \right]_{C^{\alpha}_{\mathrm{int}}(B_{2\delta_K})},$$

where the last inequality holds if ϵ is small enough.

We may also use the the interpolation inequality to estimate the lower order terms:

$$C \left(\sum_{i=1}^{n} \|b_i(x)\|_{C^{\alpha}(\Omega)} \left\| \sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i} \right\|_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c\|_{C(\Omega)} \|u(x)\|_{C^{\alpha}_{\mathrm{int},(2)}(B_{2\delta_K})} \right) \leq \tag{2.6}$$

$$\leq C_L C_{\epsilon} \delta_K^2 \| D^2 u \|_{C_{\mathrm{int}}(B_{2\delta_K})} + \frac{\delta_K^2}{4} \left[D^2 u \right]_{C^{\alpha}_{\mathrm{int}}(B_{2\delta_K})}.$$

Using (2.5) and (2.6) in (2.4) we can deduce that, for a somewhat larger C_L,

$$\begin{aligned} &\delta_K^2 \|D^2 u\|_{C^\alpha_{\text{int}}(B_{2\delta_K}(x^k))} \leq \\ &\leq C_L \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)} \right) + \\ &+ \frac{3\delta_K^2}{4} \|D^2 u\|_{C^\alpha_{\text{int}}(B_{2\delta_K}(x^k))}. \end{aligned} \tag{2.7}$$

Rearranging terms in (2.7) implies the statement in step 2.

Step 3: *Proof of the Theorem.*

Since the balls $B_{\delta_K}(x^k)$ cover K it follows directly from step 2 and that $\delta_K \geq \frac{\text{dist}(K,\partial\Omega)}{4}$ that

$$\sup_K |D^2 u(x)| \leq C_{a_{ij}} \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^2}.$$

Moreover, for any two $x, y \in K$ such that $|x-y| > \frac{\text{dist}(K,\partial\Omega)}{8}$ it follows that $x, y \in B_{\delta_k}(x^k)$ for some ball and thus

$$\frac{\left|D^2 u(x) - D^2 u(y)\right|}{|x-y|^\alpha} \leq \frac{2}{|x-y|^\alpha} \sup_K |D^2 u(x)| \leq C \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^{2+\alpha}}.$$

So we only need to estimate $\frac{\left|D^2 u(x) - D^2 u(y)\right|}{|x-y|^\alpha}$ for $|x-y| \leq \frac{\text{dist}(K,\partial\Omega)}{8} \leq \frac{\min_k(\delta_K)}{2}$. But if $|x-y| \leq \frac{\min_k(\delta_K)}{2}$ then there exists a ball $B_{\delta_K}(x^k)$ such that $x, y \in B_{\delta_K}(x^k)$ so we may use step 2 again and conclude that

$$\frac{\left|D^2 u(x) - D^2 u(y)\right|}{|x-y|^\alpha} \leq C \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^{2+\alpha}}.$$

Thus it follows that

$$\text{dist}(K,\partial\Omega)^2 \|D^2 u\|_{C^\alpha_{\text{int}}(\Omega)} \leq C \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^{2+\alpha}},$$

where $C = C(n, \alpha, \Omega, \lambda, \Lambda, a_{ij})$. The Theorem follows by the interpolation inequality Proposition 4. $\square$

Appendix A

Barnach Spaces.

We will need some notation from functional analysis in order to simplify the exposition somewhat. The point of this appendix is not to cover functional analysis, which is a very large area of mathematics. But just to remind ourselves of some basic notions. We start with the following definition.

Definition 1. *We say that a set A is a linear space over $\mathbb{R}$ if*

1. *A is a commutative group. That is there is an operation "$+$" defined on $A \times A \mapsto A$ such that*
   - (a) *For any $u, v, w \in A$ the following holds: $u + v = v + u$ (addition is commutative), $(u + v) + w = u + (v + w)$ (addition is associative).*
   - (b) *There exists an element $0 \in A$ such that for all $u \in A$ we have $u + 0 = u$.*
   - (c) *For every $u \in A$ there exists an element $v \in A$ such that $u + v = 0$, we usually denote $v = -u$.*
2. *There is an operation (multiplication) defined on $\mathbb{R} \times A \mapsto A$ such that*
   - (a) *For all $a, b \in \mathbb{R}$ and $u, v \in A$ we have $a \cdot (u + v) = a \cdot u + a \cdot v$ and $(a + b) \cdot u = a \cdot u + b \cdot u$.*
   - (b) *For all $a, b \in \mathbb{R}$ and $u \in A$ we have $(ab) \cdot u = a \cdot (b \cdot u)$.*

Examples: **1:** The most obvious example is if $A = \mathbb{R}^n$ and "$+$" is normal vector addition and "$\cdot$" is normal multiplication by a real number.

2: Another example that will be much more important to us is if A is a set of functions, say the set of functions with two continuous derivatives on Ω. Clearly all the above assumptions are satisfied for twice continuously differentiable functions if we interpret "$+$" and "$\cdot$" as the normal operations.

Many linear spaces satisfies another important structure: that we can measure distances. Distances allow us to talk about convergence and to do analysis. We will only be interested in spaces where we have a norm.

Definition 2. *A norm $\|\cdot\|$ on a linear space A is a function from $A \mapsto \mathbb{R}$ such that the following axioms are satisfied:*

1. *For any $u \in A$ we have $\|u\| \geq 0$ with equality if and only if $u = 0$ (The Positivity Axiom).*
2. *For any $u, v \in A$ we have $\|u + v\| \leq \|u\| + \|v\|$ (The Triangle Inequality).*
3. *For any $u \in A$ and $a \in A$ we have $\|a \cdot u\| = |a|\|u\|$ (The Homogeneity Axiom).*

If a linear space A has a norm we say that A is a normed linear space, or just a normed space.

Examples: 1: The linear space $\mathbb{R}^n$ is a normed space with norm $\|(u_1, ..., u_n)\| = (u_1^2 + u_2^2 + ... + u_n^2)^{1/2}$.

2: The set of continuous functions on $[0, 1]$ is a normed space under the norm

$$\|u\| = \int_0^1 |u(x)|dx.$$

3: If we define

$$\|u\|_{C^2(\Omega)} = \sup_{x\in\Omega} |u(x)| + \sup_{x\in\Omega} |\nabla u(x)| + \sup_{x\in\Omega} |D^2 u(x)|, \tag{A.1}$$

Then the set of two times continuously differentiable functions $u(x)$ on Ω for which $\|u\|_{C^2(\Omega)}$ is finite forms a normed space: $C^2(\Omega)$. Notice that $\frac{1}{x} \notin C^2(0, 1)$ even though $\frac{1}{x}$ is continuous with continuous derivatives on $(0, 1)$.

The final property that we need in our function-spaces is completeness.

Definition 3. *Let A be a normed linear space. Then we say that A is complete if every Cauchy sequence $u^j \in A$ converges in A.*

Remember that we say that $u^j \in A$ is a Cauchy sequence if there for every $\epsilon > 0$ exists a N_ϵ such that $\|u^j - u^k\| < \epsilon$ for all $j, k > N_\epsilon$. So if A is complete and u^j is a Cauchy sequence in A then there should exist an element $u^0 \in A$ such that $\lim_{j\to\infty} \|u^j - u^0\| = 0$.

Examples: 1: It is an easy consequence of the the Bolzano-Weierstrass theorem that $\mathbb{R}^n$ is complete. In particular, every Cauchy sequence is bounded. Therefore the Bolzano-Weierstrass theorem implies that it has a convergent subsequence. That the Cauchy condition implies that the entire sequence converges to the same limit is easy to see.

2: The space of continuous functions on $[0, 1]$ with norm $\|u\| = \int_0^1 |u(x)|dx$ is not complete. For instance if

$$u^j(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{j} \\ \frac{j}{2}\left(x - \left(\frac{1}{2} - \frac{1}{j}\right)\right) & \text{if } \frac{1}{2} - \frac{1}{j} < x < \frac{1}{2} + \frac{1}{j} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{j} \leq x \leq 1 \end{cases}$$

then u^j is continuous and forms a Cauchy sequence. However the limit is clearly

$$u^0(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

But u^0 is not continuous and therefore not in the space of continuous functions on $[0,1]$. Therefore that space is not complete.

However, if we consider the space $C([0,1])$ of continuous functions with norm

$$\|u\|_{C([0,1])} = \sup_{x\in[0,1]} |u(x)|$$

then we get a complete space. This since the limit $\lim_{j\to\infty} u^j(x)$ is uniform and continuity is preserved under uniform limits.

It is important to notice that the properties of the space is dependent on the norm. Continuous functions with an integral are not complete, but continuous spaces with a supremum norm are complete.

3: The space $C^2(\Omega)$ with norm defined by the supremum as in (A.1) is also a complete space.

Clearly, in order to do analysis on a linear space it is desirable that the linear space is complete. We therefore make the following definition.

Definition 4. *We call a complete linear space is a Banach space.*

A.1 Banach spaces and PDE.

Banach spaces helps us to formulate questions in PDE in a new way.

The initial way to view a PDE is to view it point-wise. That is, for the Laplace equation for instance, we think of a solution as twice differentiable function $u(x)$ that should satisfy $\sum_{i=1}^n \frac{\partial^2 u(x)}{\partial x_i^2} = f(x)$ at every point $x \in \Omega$. There is nothing wrong with this viewpoint, and as a matter of fact everything we do in Banach spaces will depend on results we derived by using this point of view. However, as the equations becomes more complicated it is reasonable to look for a simplified conceptualization of what a PDE is. By formulating a PDE as a problem in Banach spaces we are able to leave the point-wise viewpoint behind and consider the PDe as a mapping between Banach spaces.

Let us consider a function $u \in C^2(\Omega)$ and we let

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x)$$

be an elliptic partial differential operator with continuous coefficients.[1] Then for any $u \in C^2(\Omega)$ we clearly have that $Lu(x) \in C(\Omega)$.

[1] We already know that it is more natural to consider PDE with Hölder continuous coefficients. But it is enough to have continuous coefficients for us to introduce the next idea.

We can thus view the partial differential operator L as a map between Banach spaces: $L : C^2(\Omega) \mapsto C(\Omega)$. That is, for every $u \in C^2(\Omega)$ there exists an $f \in C(\Omega)$ such that $Lu(x) = f(x)$. Similarly, L maps the subset

$$C_g^2(\Omega) = \{u \in C^2(\Omega);\ u(x) = g(x) \text{ on } \partial\Omega\} \subset C^2(\Omega)$$

into $C(\Omega)$.

Solving the PDE

$$\begin{array}{ll} Lu(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array}$$

for a given $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ is therefore the same as finding an inverse L^{-1} of the mapping $L : C_g^2(\Omega) \mapsto C(\Omega)$. If such a mapping exists then the solution is given by $u(x) = L^{-1}f(x)$.

There are several reasons to change the re-conceptualize of a problem in mathematics. One reason is that changing the point of view might clarify a difficult concept, simplify statements or show that several problems have a similar underlying structure[2]. The most important reason to change the point of view on a subject is however that one might be able to use different techniques and prove new results in the new conceptualization.

In this section we will only reformulate some of our results in this new language and fix some notation. In later chapters we will prove some fixed point theorems[3] in Banach spaces that will help us to prove existence of solutions to PDE with variable coefficients.

Example: In Theorem 1 in Chapter 2 (in the first part of these lecture notes) we proved that if $f \in C_c^\alpha(\mathbb{R}^n)$ then

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi, \tag{A.2}$$

where $N(x)$ is the Newtonian kernel, solves $\Delta u(x) = f(x)$. Using the Liouville Theorem it is easy to see that the function $u(x)$ is the only solution to $\Delta u(x) = f(x)$ that tends to zero as $x \to \infty$.

If we consider Δ as an operator

$$\Delta :\ C_0^2(\mathbb{R}^n) = \{u \in C^2(\mathbb{R}^n);\ \lim_{x\to\infty} u(x) = 0\} \mapsto C(\mathbb{R}^n).$$

Then Theorem 1 actually shows that the inverse of the Laplacian, Δ^{-1}, is well defined on $C_c^\alpha(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ and given by (A.2).

Example: We know that Δ is does not have a well defined inverse from $C_c(\mathbb{R}^n)$ to $C^2(\mathbb{R}^n)$ since there are functions $u \notin C^2(\mathbb{R}^n)$ with $\Delta u \in C_c(\mathbb{R}^n)$, see exercise 3 in the first part of these notes.

[2]For instance, viewing a PDE as a mapping between linear spaces highlights a similarity between PDE and linear algebra that might not be so easy to see otherwise.

[3]Specifically, the contraction mapping principle that we will use to develop a technique called *the method of continuity*.

Based on the above two examples it is a reasonable question to ask between what spaces does Δ have an inverse? Or more generally, when does a variable coefficient PDE $L\cdot$ have an inverse. In the next section we will introduce some Banach spaces that we know are of importance in inverting PDE.

A.2 Some Banach spaces that are important for PDE.

We already know that the Hölder spaces $C^{k,\alpha}(\Omega)$ are important in PDE theory.

Definition 5. *Given a domain Ω and u a k-times continuously differentiable function on Ω we will use the notation, for $k \in \mathbb{N}$ and $\alpha \in [0,1]$,*

$$\|u\|_{C^{k,\alpha}(\Omega)} = \sum_{j=1}^{k} \sup_{x\in\Omega} |D^j u(x)| + \sup_{x,y\in\Omega} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha}.$$

Furthermore, we let $C^{k,\alpha}(\Omega)$ denote the set of all two times differentiable functions for which $\|u\|_{C^{k,\alpha}(\Omega)} < \infty$.

When $\alpha = 0$ we will disregard α and the last term in the definition of $\|u\|_{C^{k,\alpha}(\Omega)}$ and write

$$\|u\|_{C^{k,\alpha}(\Omega)} = \|u\|_{C^{k}(\Omega)} = \sum_{j=1}^{k} \sup_{x\in\Omega} |D^j u(x)|,$$

and when $k = 0$ and $\alpha \in (0,1)$ we will write $\|u\|_{C^{0,\alpha}(\Omega)} = \|u\|_{C^{\alpha}(\Omega)}$.

It is easy to that the space $C^{k,\alpha}(\Omega)$ is a Banach space.

Lemma 1. *The space $C^{k,\alpha}(\Omega)$ is a Banach space with the norm $\|u\|_{C^{k,\alpha}(\Omega)}$.*

Proof: It is trivial to verify that $C^{k,\alpha}(\Omega)$ is a linear space and that $\|u\|_{C^{k,\alpha}}(\Omega)$ is a norm. That $C^{k,\alpha}(\Omega)$ is complete follows by the Arzela-Ascoli Theorem. □

It is quite often that we only need information about the Hölder continuity, we will therefore define the semi-norm[4]

$$[u]_{C^{\alpha}(\Omega)} = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

We have also seen that the $C^{k,\alpha}(\Omega)$ space is not always suitable for expressing our theorems. We will therefore use introduce the alternative norms $\|u\|_{C^{k,\alpha}_{\mathrm{int}}(\Omega)}$ and $\|u\|_{C^{k,\alpha}_{\mathrm{int},(l)}(\Omega)}$ that we use in our interior estimates.

[4]A semi-norm is satisfies all the requirements for a norm except that $\|u\| = 0 \Leftrightarrow u = 0$.

Definition 6. *For any $k-$times continuously differentiable function $u(x)$ defined on a domain Ω we denote by $\|u\|_{C^{k,\alpha}_{int}(\Omega)}$ the least constant Γ such that*

$$\sum_{j=0}^{k}\left(dist\,(K,\partial\Omega)^j \sup_{x\in K}|D^j u(x)|\right)+dist\,(K,\partial\Omega)^{k+\alpha}\sup_{x,y\in K}\frac{|D^k u(x)-D^k u(y)|}{|x-y|^\alpha}\leq\Gamma$$

for all compact sets $K\subset\Omega$.

More generally, we will define $\|u\|_{C^{k,\alpha}_{int,(l)}(\Omega)}$ to be the least constant Γ such that

$$\sum_{j=0}^{k}\left(dist\,(K,\partial\Omega)^{j+l} \sup_{x\in K}|D^j u(x)|\right)+dist\,(K,\partial\Omega)^{k+l+\alpha}\sup_{x,y\in K}\frac{|D^k u(x)-D^k u(y)|}{|x-y|^\alpha}\leq\Gamma$$

for all compact sets $K\subset\Omega$.

Furthermore we will denote by $C^{k,\alpha}_{int}(\Omega)$ and $C^{k,\alpha}_{int,(l)}(\Omega)$ the Banach spaces of $k-$times continuously differentiable functions for which the norms $\|u\|_{C^{k,\alpha}_{int}(\Omega)}$ and $\|u\|_{C^{k,\alpha}_{int,(l)}(\Omega)}$ are bounded.

It is easy to see that $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ are Banach spaces with their respective norms.

The norms of the spaces $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ controls the functions in the interior of Ω. In particular if $u\in C^{k,\alpha}_{\text{int},(l)}(\Omega)$ then $u\in C^{k,\alpha}(K)$ for any compact set $K\subset\Omega$. However, the norm $\|u\|_{C^{k,\alpha}(K)}$ will depend on the distance $\text{dist}(K,\partial\Omega)$ and in general functions in $C^{k,\alpha}_{\text{loc},(l)}(\Omega)$ will have infinite $C^{k,\alpha}(\Omega)$ norm. Some examples might clarify the situation.

Examples: **1.** Consider $u(x)=\sin\left(\ln\left(\frac{1}{x}\right)\right)$ defined on $(0,1/2)$. Clearly $u(x)$ is bounded and continuous so $u(x)\in C(0,1/2)$. However, $u\notin C^1(0,1/2)$ since $Du(x)=-\frac{1}{x}\cos\left(\ln\left(\frac{1}{x}\right)\right)$ which isn't bounded. But $u(x)\in C^1_{\text{int}}(0,1/2)$ since for any compact set $K=[\kappa,1/2-\kappa]\subset(0,1/2)$ we have

$$\sup_{x\in K}|u(x)|+\kappa\sup_{x\in K}|Du(x)|\leq 1+\kappa\sup_{x\in[\kappa,1/2-\kappa]}\left|\frac{1}{x}\cos\left(\ln\left(\frac{1}{x}\right)\right)\right|\leq 2.$$

Thus $\|u\|_{C^1_{\text{int}}(0,1/2)}=2$.

2: Let $u(x)=\frac{1}{1-x^2}$ be defined on $(-1,1)$. Then $u(x)$ is unbounded so $u\notin C^{k,\alpha}(-1,1)$ for any k or α.

However, $u\in C^{1,\alpha}_{\text{int},(1)}(-1,1)$ since for any compact set $K=[-1+\kappa,1-\kappa]$ we have

$$\|u\|_{C^{1,\alpha}_{\text{int},(1)}}=\kappa\sup_{x\in K}|u(x)|+\kappa^2\sup_{x\in K}|Du(x)|+\kappa^{2+\alpha}\sup_{x,y\in K}\frac{|Du(x)-Du(y)|}{|x-y|^\alpha}<\infty$$

where the upper bound is independent of $\kappa\in(0,1)$.

Observe that the norm on $C^{k,\alpha}_{\text{int},(l)}(-1,1)$ allows the function and its derivative to tend to infinity at the boundary of Ω. The parameter l determines how fast the function and its derivatives may go to infinity. For instance the above function $u \in C^{k,\alpha}_{\text{int},(l)}(-1,1)$ for any $l \geq 1$ but not for any $l < 1$.

It is important to realize that these norms, even though they appear to be artificial, they are natural. For instance we may formulate the interior regularity result for harmonic functions as:

Proposition 2. *Let Ω be a domain and assume that $u(x)$ is a solution to*

$$\Delta u(x) = f(x) \qquad \textit{in } \Omega$$

assume furthermore that $|u| \leq M$ in Ω and that $f \in C^{\alpha}_{int,(2)}(\Omega)$ then there exists a constant $C_{n,\alpha}$ such that

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \leq C_{n,\alpha} \left(\|f\|_{C^{\alpha}_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)} \right). \tag{A.3}$$

The proof of Proposition 2 is a direct consequence of Proposition 1 in the 5th part of these lecture notes together with an interpolation inequality that we will prove in the next appendix. Notice that the norms $\|u\|_{C^{2,\alpha}_{\text{int}}(\Omega)}$ and $\|f\|_{C^{\alpha}_{\text{int},(2)}(\Omega)}$ appears in the statement - and that these norms makes the statement of the Proposition much more compact than the formulation of Proposition 1 in the fifth part of these notes. The norms are natural in the sense that (A.3) is optimal and we can not prove a stronger statement without adding further assumptions on the boundary data of u and on the geometry of Ω.

Remark on scaling: One heuristic way to see that (A.3) is natural is to consider the "scaling" of the estimate. Since $\Delta u(x)$ involves two derivatives it is natural that if $\Delta u = f$ then u should have two more derivatives than f. This explains that we have a $(2,\alpha)$ norm on the right hand side in (A.3) whereas the left hand side is only a Hölder $\alpha-$norm. Since we are not making any assumptions on the boundary data of u in Proposition 2 we can not expect the derivatives of u to be bounded - in particular if the boundary data is discontinuous at $x^0 \in \partial\Omega$ then u can not have any continuous extension to $\overline{\Omega}$. So the best estimate we can hope for is an estimate that allows $|\nabla u(x)|$ and $|D^2 u(x)|$ to tend to infinity as $x \to \partial\Omega$. This explains why we have the *"int"* in the $C^{2,\alpha}_{\text{int}}(\Omega)-$norm in (A.3).

The difference between the $C^{2,\alpha}(\Omega)$ and the $C^{2,\alpha}_{\text{int}}(\Omega)-$norm is that the latter norm allows

$$|\nabla u(x)| \approx \text{dist}(x, \partial\Omega)^{-1}, \tag{A.4}$$

$$|D^2 u(x)| \approx \text{dist}(x, \partial\Omega)^{-2} \tag{A.5}$$

and

$$\sup_{x,y \in K} \frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^{\alpha}} \approx \text{dist}(\{x,y\}, \partial\Omega)^{-2-\alpha} \tag{A.6}$$

whereas the $C^{2,\alpha}(\Omega)$-norm requires uniform bounds in the entire domain Ω. To see that the exponents -1, -2 and $-2-\alpha$ in (A.4), (A.5) and (A.6) are natural we rescale the equation. We use the estimate (A.4) as an illustration. Assume that $\text{dist}(x^0, \partial\Omega) = 2r$ then the function $v(x) = u(rx + x^0)$ will solve

$$\Delta v(x) = r^2 f(rx + x^0) \qquad \text{in } B_2(0),$$

since

$$\sum_{i=1}^n \frac{\partial^2 v(x)}{\partial x_i^2} = \sum_{i=1}^n \left(r^2 \frac{\partial^2 u(rx + x^0)}{\partial x_i^2} \right) = r^2 \Delta u(rx + x^0) = r^2 f(rx + x^0). \quad \text{(A.7)}$$

Since $\sup_{B_2(0)} |v| \le \sup_\Omega |u|$ we can conclude that $|\nabla v(0)|$ is bounded independently of r. But $|\nabla v(0)| = r|\nabla u(x^0)|$ and if $r|\nabla u(x^0)|$ is bounded independently of $r \approx \text{dist}(x^0, \partial\Omega)$ then $|\nabla u(x)| \approx \text{dist}(x, \partial\Omega)^{-1}$ which is what what (A.4) states. If you consider the proof of Proposition 1 (in part 5 of the lecture notes) again you will see that that is exactly how we prove the estimates.

Finally, we need to say something about the $l = 2$ in the $\|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}$–norm of (A.3). But we see directly from the scaling in (A.7) that $l = 2$ is the optimal l since if $|f(x^0)| \approx \text{dist}(x^0, \partial\Omega)^{-2}$ (that is the growth of f allowed by the norm $\|f\|_{C^\alpha_{\text{int},(l)}(\Omega)}$ with $l = 2$) then the right hand side in (A.7) is bounded since $r \approx \text{dist}(x^0, \partial\Omega)$.

Further properties of the Hölder spaces: In addition to being a Banach space the Hölder spaces $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ also have a multiplication defined[5]: if $\phi(x), \varphi(x) \in C^{k,\alpha}(\Omega)$ then $\phi(x) \cdot \varphi(x) \in C^{k,\alpha}(\Omega)$ (and similarly for $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$).

We will only prove this for $k = 0$, the general case is an easy consequence of this and the product rule for the derivative.

Proposition 3. *Assume that* $\phi(x), \varphi(x) \in C^\alpha(\Omega)$ *then* $\phi(x) \cdot \varphi(x) \in C^{k,\alpha}(\Omega)$ *and*

$$[\phi \cdot \varphi]_{C^\alpha(\Omega)} \le \left(\|\phi\|_{C(\Omega)} [\varphi]_{C^\alpha(\Omega)} + \|\varphi\|_{C(\Omega)} [\phi]_{C^\alpha(\Omega)} \right). \qquad \text{(A.8)}$$

Proof: The proof uses the same trick as the proof of the multiplication rule for differentiation. In particular, we may estimate

$$|\phi(x)\varphi(x) - \phi(y)\varphi(y)| = |(\phi(x)\varphi(x) - \phi(x)\varphi(y)) - (\phi(y)\varphi(y) - \phi(x)\varphi(y))| \le$$
$$\le |\phi(x)|\, |\varphi(x) - \varphi(y)| + |\varphi(y)|\, |\phi(y) - \phi(x)| \le \qquad \text{(A.9)}$$
$$\le \|\phi(x)\|_{C(\Omega)} |\varphi(x) - \varphi(y)| + \|\varphi(y)\|_{C(\Omega)} |\phi(y) - \phi(x)|,$$

where the last inequality follows since $\|\phi(x)\|_{C(\Omega)} = \sup_{x\in\Omega} |\phi(x)|$ by definition.

[5]The technical term is that $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ are *algebras* over $\mathbb{R}$.

If we divide both sides in (A.9) by $|x-y|^\alpha$ it follows that

$$\frac{|\phi(x)\varphi(x)-\phi(y)\varphi(y)|}{|x-y|^\alpha} \leq \|\phi(x)\|_{C(\Omega)} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} + \|\varphi(y)\|_{C(\Omega)} \frac{|\phi(y)-\phi(x)|}{|x-y|^\alpha} \leq$$

$$\leq \left(\|\phi\|_{C(\Omega)}[\varphi]_{C^\alpha(\Omega)} + \|\varphi\|_{C(\Omega)}[\phi]_{C^\alpha(\Omega)}\right),$$

by the definition of $[\varphi]_{C^\alpha(\Omega)}$ and $[\phi]_{C^\alpha(\Omega)}$. Taking the supremum over $x, y \in \Omega$ yields the result. $\square$

Appendix B

Interpolation inequalities

An interpolation inequality is exactly what it sounds like. Given two inequalities we might derive a third inequality that somehow lies between the other two. In this chapter we will show that if the second derivatives and the function value (zeroth order derivatives) of u is bounded, then the first derivatives are bounded as well. We will only prove the two simple interpolation inequalities that we need

Proposition 4. [Interpolation inequality] *Suppose that $u \in C(\Omega)$ then:*

1. *If $D^2u \in C_{int,(2)}(\Omega)$ then, for any $\epsilon > 0$, there exists a C_ϵ such that the following inequality holds*

$$\|\nabla u\|_{C_{int,(1)}} \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon \|D^2 u\|_{C_{int,(2)}}. \tag{B.1}$$

2. *If $[D^2u]_{C^{\alpha}_{int,(2)}(\Omega)}$ is bounded then, for any $\epsilon > 0$, there exists a C_ϵ such that the following inequality holds*

$$\|D^2 u\|_{C_{int,(2)}} \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon [D^2 u]_{C^{\alpha}_{int,(2)}(\Omega)}. \tag{B.2}$$

3. *The same is true without the "int$''$ and (l) in the norms.*

Remark on the proposition. The proposition might seem to be very abstract (in particular if one is unused to the rather intricate definitions of the norms). But what it states is that it is enough to control $\|u\|_{C(\Omega)}$ and $\|D^2u\|_{C_{\text{int},(2)}}$ in order to control the norm

$$\|u\|_{C^2_{\text{int}}(\Omega)} = \|u\|_{C(\Omega)} + \|\nabla u\|_{C_{\text{int},(1)}} + \|u\|_{C(\Omega)} + \|D^2 u\|_{C_{\text{int},(2)}}.$$

Similarly, $\|u\|_{C(\Omega)}$ and $[D^2u]_{C^{\alpha}_{\text{int},(2)}(\Omega)}$ controls the norm $\|u\|_{C^{2,\alpha}_{\text{int}}(\Omega)}$.

Proof: We will only prove the first two points since the third point is analogous.

To show (B.1) we let $x^0 \in \Omega$. We need to show that

$$\operatorname{dist}(x^0, \partial\Omega)|\nabla u(x^0)| \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon \|D^2 u\|_{C_{\mathrm{int},(2)}}. \tag{B.3}$$

If we can show (B.3) then (B.1) follows by taking the supremum over all $x^0 \in \Omega$.

If we let $4d = \operatorname{dist}(x^0, \partial\Omega)$ then

$$\sup_{B_d(x^0)} |D^2 u(x)| \leq \frac{C}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}}$$

and from Taylors Theorem we can conclude that, for any $0 \leq t \leq d$,

$$\inf_{B_t(x^0)} |\xi \cdot \nabla u(x)| \geq |\nabla u(x^0)| - \frac{Ct}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}}, \tag{B.4}$$

where $\xi = \frac{\nabla u(x^0)}{|\nabla u(x^0)|}$.

Now for any $y^1, y^2 \in B_d(x^0)$ such that $y^2 = y^1 + s\xi$ there exists, by the mean value theorem a $z \in B_d(x^0)$ on the line between y^1 and y^2 such that

$$2 \sup_{B_d(x^0)} |u(x)| \geq |u(y^1) - u(y^2)| = \underbrace{|(y^2 - y^1) \cdot \nabla u(z)|}_{=|s\xi \cdot \nabla u(z)|} \geq \tag{B.5}$$

$$\geq s|\nabla u(x^0)| - \frac{Cs^2}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}},$$

where we used (B.4) with $s = t$ in the last inequality.

Rearranging (B.5) and then multiply both sides by $\frac{\operatorname{dist}(x^0, \partial\Omega)}{s}$ we see that

$$\frac{2\operatorname{dist}(x^0, \partial\Omega)}{s} \|u\|_{C(\Omega)} + \frac{Cs\operatorname{dist}(x^0, \partial\Omega)}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}} \geq \operatorname{dist}(x^0, \partial\Omega)|\nabla u(x^0)||\nabla u(x^0)|.$$

But $4d = \operatorname{dist}(x^0, \partial\Omega)$ and $s > 0$ is arbitrary so we can choose $s = c\epsilon d$ for an appropriate $c > 0$ and conclude that

$$\frac{C}{\epsilon} \|u\|_{C(\Omega)} + \epsilon \|D^2 u\|_{C_{\mathrm{int},(2)}} \geq \operatorname{dist}(x^0, \partial\Omega)|\nabla u(x^0)|.$$

This is exactly what we want to prove with $C_\epsilon = C/\epsilon$, (B.1) follows.

Next we prove (B.2).The proof is very similar to the proof of (B.1). However, we will need to use a second order Taylor expansion instead of a first order expansion. As before we fix an $x^0 \in \Omega$ and set $4d = \operatorname{dist}(x^0, \partial\Omega)$.

We aim to show that for $x^0 \in \Omega$

$$\operatorname{dist}(x^0, \partial\Omega)^2 |D^2 u(x^0)| \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon [D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)}. \tag{B.6}$$

Notice that it is enough to show that for all unit vectors η

$$\operatorname{dist}(x^0, \partial\Omega)^2 |D^2_\eta u(x^0)| \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon [D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)},$$

where $D_\eta = \eta \cdot \nabla$ is the directional derivative in the η direction. There is no loss of generality to assume that $\eta = e_1$, otherwise we may change basis for our coordinate system so that $\eta = e_1$.

Using a Taylor expansion we see that, for $y^0 = x^0 + se_1$ and $|s| \leq d$,

$$\begin{aligned}
&\left| u(y^0) - \left(u(x^0) + \frac{\partial u(x^0)}{\partial x_1}(y_1^0 - x_1^0) + \frac{1}{2}\frac{\partial^2 u(x^0)}{\partial x_1^2}(y_1^0 - x_1^0)^2 \right) \right| = \\
&= \left| u(y^0) - \left(u(x^0) + \frac{\partial u(x^0)}{\partial x_1}s + \frac{1}{2}\frac{\partial^2 u(x^0)}{\partial x_1^2}s^2 \right) \right| \leq \\
&\leq C\frac{|s|^{2+\alpha}}{d^2}[D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)}.
\end{aligned} \tag{B.7}$$

Let us, for the sake of definiteness assume that $\frac{\partial^2 u(x^0)}{\partial x_1^2} \leq 0$ then we may choose s such that $s\frac{\partial u(x^0)}{\partial x_1} \leq 0$ and conclude from (B.7) that

$$u(y^0) - u(x^0) - \frac{1}{2}\frac{\partial^2 u(x^0)}{\partial x_1^2}s^2 \leq C\frac{|s|^{2+\alpha}}{d^2}[D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)},$$

which implies that

$$\left| \frac{\partial^2 u(x^0)}{\partial x_1^2} \right| \leq \frac{4}{s^2}\|u\|_{C(\Omega)} + C\frac{|s|^\alpha}{d^2}[D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)},$$

which gives (B.6) if we choose $|s|$ small enough and that $4d = \mathrm{dist}(x^0, \partial\Omega)$. $\square$