Selected Topics in PDE part 7.

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Chapter 1

An interlude - the Need for Boundary Estimates.

So far we have proved interior estimates, that is estimates for $\|u\|_{C^{2,\alpha}_{\text{int}}}$ if u solves an elliptic PDE. Unfortunately the interior estimates are not strong enough to prove existence of solutions since they allow the second derivatives to grow line $\text{dist}(x,\partial\Omega)^{-2}$.

In order to explain this let us review our strategy for finding solutions to the equation

$$\begin{array}{ll} \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

let us for notational simplicity assume that $b_i = c = 0$. We write the equation

$$\sum_{i,j=1}^n a_{ij}(x^0)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \underbrace{\sum_{i,j=1}^n \big(a_{ij}(x^0) - a_{ij}(x)\big)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + f(x)}_{=F(x)}, \tag{1.2}$$

where we assume that $|a_{ij}(x^0) - a_{ij}(x)| < \epsilon$. Notice that if $u \in C^{2,\alpha}_{\text{int}}(\Omega)$ then the right hand side in (1.2) may grow like $|F(x)| \approx \frac{\epsilon}{\text{dist}(x,\partial\Omega)^2}$ as we approach the bundary $\partial\Omega$.

In order to show that the boundary values are obtained in (1.1) we would need to construct a barrier $w(x)$ at each boundary point $x^0 \in \Omega$. A barrier was a super-solution to the equation that satisfied $w(x^0) = 0$ and $w(x^0) > 0$ in $\overline{\Omega} \setminus \{x^0\}$. But a super-solution would have to satisfy

$$\sum_{i,j=1}^n a_{ij}(x^0)\frac{\partial^2 w(x)}{\partial x_i \partial x_j} \leq F(x).$$

And if $F(x) \approx -\frac{\epsilon}{\text{dist}(x,\partial\Omega)^2}$ then it is easy to see that we can not find a barrier in general. The easiest way to see this is to consider the one dimensional problem $\Omega = (0,1)$ and $F(x) = -\frac{\epsilon}{\text{dist}(x,\partial\Omega)^2}$ and $a_{11}(x^0) = 1$. Then the equation for the barrier reduces to

$$\begin{array}{ll} \frac{d^2w(x)}{dx^2} \leq -\frac{\epsilon}{x^2} & \text{in } (0,1) \\ w(0) = 0 & \text{and} \\ w(x) > 0 & \text{in } (0,1). \end{array}$$

But integrating this differential equation leads to $w(x) = \epsilon \ln(x) + ax + b$ for some constants $a, b \in \mathbb{R}$ which clearly can not take the value $w(0) = 0$.

The problem is that the interior estimates allow the solution to grow to fast at the boundary (that is why they are called interior). Therefore we need to prove some estimates at the boundary of the domain. It is easy to see that we can not prove that the solution to (**??**) has bounded $C^{2,\alpha}$ norm without any assumptions on the boundary and on the domain.

Example: Let $\Omega = B_1^+(0) = \{x \in B_1(0)\ x_n > 0\}$ be a domain in $\mathbb{R}^2$ and $u(x)$ be a solution to

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \Omega \\ u(x) = |x_1|^\alpha & \text{on } \partial\Omega, \end{array}$$

for some $\alpha \in (0,1)$. Such a solution exists by the Perron method. However, if $\|u\|_{C^{2,\alpha}(\Omega)} \leq C$ then we would have that $\|u(x_1,0)\|_{C^{2,\alpha}(x_1 \in (-1,1))} \leq C$. But $u(x_1,0) = |x_1|^\alpha \notin C^{2,\alpha}$ which would lead to a contradiction. We may conclude that $u \notin C^{2,\alpha}(\Omega)$. As a matter of fact, this shows that the best we can hope for is that $u \in C^\alpha(\Omega)$. This shows that we must assume that the boundary data is in $C^{2,\alpha}$ to have any hope to show that $\|u\|_{C^{2,\alpha}(\Omega)}$ is bounded.

Example: Remember that the function $u(r,\phi) = r^\alpha \sin(\alpha\phi)$ solves the Dirichlet problem

$$\begin{array}{ll} \Delta u(r,\phi) = 0 & \text{in } \{r \in (0,\infty),\ \phi \in (0,\pi/\alpha) \\ u(r,\phi) = 0 & \text{for } \phi = 0 \text{ and } \phi = \frac{\pi}{\alpha}, \end{array}$$

for $\alpha \geq \frac{1}{2}$. Notice that if $\alpha \in [1/2, 1)$ then $u \in C^\alpha \setminus C^1$. So we have harmonic functions with zero boundary data that are still not $C^{2,\alpha}$. The problem here is that the domain has a sharp corner at the origin. Apparently we need to assume something about the regularity of the domain in order to prove that the solutions are $C^{2,\alpha}(\Omega)$.

In the following chapters we will pursue estimates for the $C^{2,\alpha}-$norm for solutions to (1.1). The proofs will be quite similar to the proofs of the interior estimates. In particular, we will start to show boundary estimates for the Newtonian potential close to a part of the boundary where the boundary is assumed to be contained in a hyperplane. Then we will continue to investigate the Dirichlet problem for the laplace equation close to a boundary, again given by a hyperplane. Having those estimates at hand it is easy to show apriori estimates for solutions to the Dirichlet problem for variable coefficient PDE.

Chapter 2

Boundary regularity - The Laplace equation.

In this chapter we will investigate the boundary regularity properties for the Laplace equations close to a part of the boundary that is a hyperplane. The proof will be analogous to the interior regularity proof.

We begin by estimating the Newtonian potential in an upper half ball $B_{2R}^+(0)$. The proof consists of one major observation - that the boundary terms on the flat part of the boundary disappears in the estimate for the second derivatives for all second derivatives except $\frac{\partial^2 u(x)}{\partial x_n^2}$. But it is easy to estimate $\frac{\partial^2 u(x)}{\partial x_n^2}$ in terms of $\frac{\partial^2 u(x)}{\partial x_i^2}$, for $i = 1, 2, ..., n-1$ and $f(x)$. This since $\Delta u(x) = f(x)$ and thus $\frac{\partial^2 u(x)}{\partial x_n^2} = f(x) - \sum_{i=1}^{n-1} \frac{\partial^2 u(x)}{\partial x_i^2}$

Lemma 1. *Let $f(x) \in C^\alpha(B_{2R}^+(0))$ for some $0 < \alpha < 1$ and define*

$$u(x) = \int_{B_{2R}^+(0)} N(x-\xi) f(\xi) d\xi$$

then there exists a constant $C_{n,\alpha}$ depending only on n and α such that the following inequality holds

$$[D^2 u]_{C^\alpha(B_R^+(0))} \leq C_{n,\alpha} \left([f]_{C^\alpha(B_{2R}^+(0))} + \frac{\sup_{B_R(0)} |f(x)|}{R^\alpha} \right). \tag{2.1}$$

Proof: We have already shown, see Theorem **??** (Thm 1 in part 1 of these notes), that $\frac{\partial^2 u(x)}{\partial x_i \partial x_j}$ has the following representation formula for $x \in B_{2R}^+(0)$

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_{2R}^+(0)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x) \right) d\xi - f(x) \int_{\partial B_{2R}^+(0)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi). \tag{2.2}$$

Strictly we only proved this representation for the domain $B_{2R}(0)$ but the deduction for the upper half ball $B_{2R}^+(0)$ is exactly the same.

We will split the proof into two cases. The first case is very similar to the proof of Theorem **??** (Thm 1 part 5 of these notes) and we will only indicate the minor differences.

Case 1: *Estimates for* $\left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{C^\alpha(B_R^+(0))}$ *when* $i \neq n$ *or* $j \neq n$.

We may assume that $j \neq n$, if not then $i \neq n$ and we may use that $\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \frac{\partial^2 u(x)}{\partial x_j \partial x_i}$ to reduce to the case for $j \neq n$.

Observe that the normal $\nu = -e_n$ on $\partial B_{2R}^+(0) \cap \{x_n = 0\}$ and the boundary integral in (2.2) therefore reduces to

$$f(x)\int_{\partial B_{2R}^+(0)} \frac{\partial N(x-\xi)}{\partial x_i}\nu_j(\xi)dA(\xi) =$$

$$= f(x)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} \frac{\partial N(x-\xi)}{\partial x_i}\nu_j(\xi)dA(\xi).$$

Therefore, for $j \neq n$, the representation in (2.2) becomes

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_{2R}^+(0)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\left(f(\xi) - f(x)\right)d\xi - \tag{2.3}$$

$$-f(x)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} \frac{\partial N(x-\xi)}{\partial x_i}\nu_j(\xi)dA(\xi).$$

Notice that we do not integrate over the set $\{x_n = 0\}$ in (2.3). We may therefore form the difference

$$\left|\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right| =$$

$$= \left|\int_{B_{2R}^+(0)} N_{ij}(x-\xi)\left(f(\xi) - f(x)\right)d\xi - f(x)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} N_i(x-\xi)\nu_j dA(\xi) - \right.$$

$$\left. - \int_{B_{2R}^+(0)} N_{ij}(y-\xi)\left(f(\xi) - f(y)\right)d\xi + f(y)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} N_i(y-\xi)\nu_j dA(\xi)\right|.$$

This the integrals we estimate in the proof of Theorem **??** (Thm 1 part 5 of these notes) with the only difference that we now integrate over a smaller set $B_{2R}(0) \cap \{x_n > 0\}$ in place of $B_{2R}(0)$. But the estimates of Theorem **??** (Thm 1 part 5 of these notes) still works line for line in this case.

We may therefore conclude that, for $j \neq n$,

$$\left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{C^\alpha(B_R^+(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right). \tag{2.4}$$

Case 2: *Estimates for* $\left[\frac{\partial^2 u}{\partial x_n^2}\right]_{C^\alpha(B_R^+(0))}$.

Since $\Delta u(x) = f(x)$ it follows that

$$\frac{\partial^2 u(x)}{\partial x_n^2} = f(x) - \sum_{j=1}^{n-1} \frac{\partial^2 u(x)}{\partial x_j^2}.$$

In particular

$$\left[\frac{\partial^2 u}{\partial x_n^2}\right]_{C^\alpha(B_R^+(0))} = \left[f(x) - \sum_{j=1}^{n-1} \frac{\partial^2 u(x)}{\partial x_j^2}\right]_{C^\alpha(B_R^+(0))} \leq$$

$$\leq [f(x)]_{C^\alpha(B_R^+(0))} + \sum_{j=1}^{n-1} \left[\frac{\partial^2 u(x)}{\partial x_j^2}\right]_{C^\alpha(B_R^+(0))} \leq$$

$$\leq [f(x)]_{C^\alpha(B_R^+(0))} + (n-1)C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right),$$

where we used the triangle inequality in the first inequality and (2.4) in the last inequality.

If we redefine $C_{n,\alpha}$ to $1 + (n-1)C_{n,\alpha}$ we may conclude that

$$\left[\frac{\partial^2 u}{\partial x_n^2}\right]_{C^\alpha(B_R^+(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right).$$

□

Corollary 1. *Let u be as in Lemma 1 then*

$$\|u\|_{C^{2,\alpha}(B_R^+(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}^+(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f(x)\|_{C(B_{2R}^+(0))}\right).$$

Proof: By the interpolation inequality it is enough to show that

$$\|u\|_{C(B_{2R}^+(0))} \leq C_n R^2 \|f(x)\|_{C(B_{2R}^+(0))}.$$

By the definition of u we have

$$|u(x)| = \left|\int_{B_{2R}^+(0)} N(x-\xi)f(\xi)d\xi\right| \leq$$

$$\leq \|f\|_{C(B_{2R}^+(0))}\left|\int_{B_{2R}^+(0)} N(\xi)d\xi\right| \leq C_n R^2 \|f(x)\|_{C(B_{2R}^+(0))},$$

where we used the explicit formula for N in the last inequality. □

Next we estimate the solution to the Dirichlet problem up to the boundary in B_{4R}^+ with zero boundary data on $x_n = 0$. The proof uses that we may reflect the potential solution from Lemma 1 in the hyperplane $\{x_n = 0\}$, just as we reflected the Newtonian kernel in order to find a Greens function in $\mathbb{R}^n_+$. This allows us to reduce the regularity problem to the case when $f(x) = 0$. An odd reflection in $x_n = 0$ to the solutions with $f(0) = 0$ reduces the boundary regularity case to an interior problem.

Proposition 1. *Assume that $u \in C^2(B_{4R}^+)$ and that u solves*

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } B_{2R}^+(0) \\ u(x) = 0 & \text{on } B_{4R}(0) \cap \{x_n = 0\}, \end{array}$$

where $f \in C^\alpha(B_{4R}^+(0))$ for some $\alpha \in (0,1)$.

Then there exists a constant $C_{n,\alpha}$ depending only on n and α such that

$$[u]_{C^{2,\alpha}(B_R^+(0))} \leq \tag{2.5}$$

$$\leq C_{n,\alpha}\left([f]_{C^\alpha(B_R^0(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B_{4R}^+(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B_{2R}^+(0))}\right).$$

Proof: We will write $u(x) = v(x) + h(x)$ in the ball $B_{2R}^+(0)$ where

$$\begin{array}{ll} \Delta v(x) = f(x) & \text{in } B_{2R}^+(0) \\ v(x) = 0 & \text{on } B_{2R}(0) \cap \{x_n = 0\} \end{array}$$

and

$$\begin{array}{ll} \Delta h(x) = h(x) & \text{in } B_{2R}^+(0) \\ h(x) = 0 & \text{on } B_{2R}(0) \cap \{x_n = 0\} \\ h(x) = u(x) - v(x) & \text{on } \partial B_{2R}(0) \cap \{x_n > 0\}. \end{array} \tag{2.6}$$

We need to estimate the $C^{2,\alpha}(B_R^+(0))-$norms of $v(x)$ and $h(x)$ in turn.

Step 1: *Construction of and estimates for $v(x)$.*

We may define the reflection of $f(x)$ in $C^\alpha(B_{4R}(0))$ according to

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x_n \geq 0 \\ f(x_1, x_2, .., x_{n-1}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Then $\hat{f} \in C^\alpha(B_{4R}(0))$ and $\|\hat{f}\|_{C^\alpha(B_{4R}(0))} = \|f\|_{C^\alpha(B_{4R}^+(0))}$.

Now define

$$\hat{v}(x) = \int_{B_{4R}(0)} N(x-\xi)\hat{f}(\xi)d\xi \quad \text{for } x \in B_{4R}(0)$$

and

$$\check{v}(x) = \int_{B_{4R}^+(0)} N(x-\xi)f(\xi)d\xi \quad \text{for } x \in B_{4R}^+(0).$$

From Theorem **??** (Thm 1 part 5 of these notes) we derive that

$$[D^2\hat{v}]_{C^\alpha(B_2R(0)} \leq C_{\alpha,n}\left([f]_{C^\alpha(B_{4R}(0))} + \frac{\sup_{B_{2R}(0)}|f(x)|}{R^\alpha}\right),$$

and similarly from Lemma 1 we derive that

$$[D^2\check{v}]_{C^\alpha(B^+_{2R}(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B^+_{4R}(0))} + \frac{\sup_{B_{2R}(0)}|f(x)|}{R^\alpha}\right).$$

In particular we may conclude that $v(x) = 2\check{v}(x) - \hat{v}(x)$ satisfies the same estimate (possibly with a larger constant)

$$[D^2 v]_{C^\alpha(B^+_{2R}(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B^+_{4R}(0))} + \frac{\sup_{B_{2R}(0)}|f(x)|}{R^\alpha}\right).$$

We claim that $v(x_1, x_2, ..., x_{n-1}, 0) = 0$. This follows easily from the symmetry of the Newtonian kernel:

$$N(x_1-\xi_1, ..., x_{n-1}-\xi_{n-1}, x_n-\xi_n) = N(x_1-\xi_1, x_2-\xi_2, ..., x_{n-1}-\xi_{n-1}, -x_n+\xi_n),$$

since $N(x-\xi)$ only depends on $|x-\xi|$.

Therefore, if $x_n = 0$,

$$\hat{v}(x) = \int_{B_{4R}(0)} N(x-\xi)\hat{f}(\xi)d\xi =$$

$$= \int_{B_{4R^+}(0)} N(x-\xi)\hat{f}(\xi)d\xi + \int_{B_{4R^-}(0)} N(x-\xi)\hat{f}(\xi)d\xi =$$

$$= 2\int_{B_{4R^+}(0)} N(x-\xi)\hat{f}(\xi)d\xi = 2\check{v}(x).$$

So if $x_n = 0$ then $v(x) = 2\check{v}(x) - \hat{v}(x) = 0$ as claimed.

Step 2: *Construction of and estimates for $h(x)$.*

The function $h(x) = u(x) - v(x)$ so we only need to estimate its $C^{2,\alpha}$ norm.

We will do that by considering the odd reflection of $h(x)$ - which we will show is harmonic in $B_{2R}(0)$ - together with interior estimates for harmonic functions. In particular that we may estimate the $C^3(B_R(0))-$norm of a harmonic functions by its $C(B_{2R})-$norm.

We need to estimate $\|h\|_{C(B^+_{2R}(0))}$ which, by the maximum principle, is the same as estimating

$$\sup_{\partial B^+_{2R}(0)}|h(x)| = \sup_{\partial B^+_{2R}(0)}|u(x)-v(x)| \leq \sup_{\partial B^+_{2R}(0)}|u(x)| + \sup_{\partial B^+_{2R}(0)}|v(x)|.$$

The supremum of u appears in the right hand side of (2.5) so we only need to estimate $\sup_{\partial B^+_{2R}(0)} |v(x)|$. This is easily done as in the proof of Corollary 1. In particular,

$$|v(x)| \le 2\left|\int_{B^+_{4R}(0)} N(x-\xi)f(\xi)d\xi\right| + \left|\int_{B_{4R}(0)} N(x-\xi)\hat{f}(\xi)d\xi\right| \le$$

$$\le 3\|f\|_{C(B^+_{4R}(0))}\int_{B_{4R}} |N(\xi)|\, d\xi \le C_n\|f\|_{C(B^+_{4R}(0))}R^2.$$

Therefore

$$\|h\|_{C(B^+_{2R}(0)} \le \|u\|_{C_{B^+_{2R}(0)}} + C_n\|f\|_{C(B^+_{4R}(0))}R^2.$$

Consider the odd reflection of h on $\partial B_{2R}(0)$:

$$\hat{h}(x) = \begin{cases} h(x) & \text{if } x_n \ge 0 \text{ and } x \in \partial B_{2R}(0) \\ -h(x) & \text{if } x_n < 0 \text{ and } x \in \partial B_{2R}(0). \end{cases}$$

Furthermore we let g solve the Dirichlet problem

$$\begin{array}{ll} \Delta g(x) = 0 & \text{in } B_{2R}(0) \\ g(x) = \hat{h}(x) & \text{on } \partial B_{2R}(0). \end{array} \tag{2.7}$$

Then, since $g(x)$ is uniquely determined by (2.7) and since $\hat{h}(x)$ is odd in x_n, it follows that $g(x)$ is an odd function in x_n. That is $g(x_1, x_2, ..., 0) = 0$ and therefore $g(x)$ solves (2.6). Uniqueness for the Dirichlet problem implies that $h(x) = g(x)$ in $B^+_{2R}(0)$.

Now, since $g(x)$ is harmonic in $B_{2R}(0)$ it follows that there exists a constant C_n such that

$$\|D^3h\|_{C(B^+_R(0))} = \|D^3g(x)\|_{C(B^+_R(0))} \le \|D^3g(x)\|_{C(B_R(0))} \le$$

$$\le \frac{C_n}{R^3}\|g\|_{C(B_{2R})} = \frac{C_n}{R^3}\|h\|_{C(B_{2R})}.$$

An application of the mean-value Theorem from calculus implies that

$$[D^2h]_{C^\alpha(B^+_R(0))} \le R^{1-\alpha}\|D^3h\|_{C(B^+_R(0))} \le \frac{C_n}{R^{2+\alpha}}\|h\|_{C(B_{2R})} \le$$

$$\le C_n\left(\left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

In particular, we have shown that

$$[D^2u]_{C^\alpha(B^+_R(0))} \le C_{n,\alpha}\left([h]_{C^\alpha(B^0_R(0))} + [h]_{C^\alpha(B^0_R(0))}\right) \le$$

$$\le C_{n,\alpha}\left([f]_{C^\alpha(B^0_R(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

□

Corollary 2. *Under the assumptions of Proposition 1 we have the estimate*

$$\|u\|_{C^{2,\alpha}(B_R^+(0))} \leq \tag{2.8}$$

$$\leq C_{n,\alpha}\left([f]_{C^\alpha(B_R^0(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B_{4R}^+(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B_{2R}^+(0))}\right).$$

Proof: We only need to estimate $\|\nabla u\|_{C(B_R^+(0))}$ and $\|D^2 u\|_{C(B_R^+(0))}$. However, that can be done by an interpolation inequality. □

We end this chapter with a proposition for constant coefficient PDE. The proof is, as it was for the interior case, based on a change of variables that reduces the PDE to the Laplacian.

Proposition 2. *Let $u(x)$ be a solution to the constant coefficient elliptic PDE*

$$\begin{array}{ll} \sum_{i,j=1}^n a_{ij}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) & \text{in } B_{2R}^+(0) \\ u(x) = 0 & \text{on } \partial\Omega \cap \{x_n = 0\}, \end{array} \tag{2.9}$$

where $a_{ij} = a_{ji}$ satisfies the ellipticity condition for all $\xi in \mathbb{R}^n$ and some $\lambda, \Lambda > 0$

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2.$$

Then there exists a constant $C_{\lambda,\Lambda,n,\alpha} > 0$ such that

$$\|u\|_{C^{2,\alpha}(B_R^+(0))} \leq \tag{2.10}$$

$$\leq C_{\lambda,\Lambda,n,\alpha}\left([f]_{C^\alpha(B_R^0(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B_{4R}^+(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B_{2R}^+(0))}\right).$$

Proof: As in the proof of the interior estimates for constant coefficient PDEs we make the change of variables $v(x) = u(Px)$ where P is chosen such that $P^T A P = I$. Notice that the linear transformation P will map $\{x_n = 0\}$ unto a hyperplane that we may assume (possibly after a rotation of the coordinates) to be $\{x_n = 0\}$. We may thus apply Corollary 2 on $v(x)$ and then use $u(x) = v(P^{-1}x)$ to derive the desired estimates for u. For further details see the proof of Proposition **??** (Part 6 of these notes).

Chapter 3

Boundary Regularity - Variable Coefficient Equations.

In this chapter we prove apriori estimates up to the boundary for general linear variable coefficient PDE for $C^{2,\alpha}$ domains. We start by showing estimates for variable coefficient equations in upper half balls $B_{2R}(0)$ and then we show that general domains with $C^{2,\alpha}$ boundaries can be reduced to this case.

3.1 Boundary Regularity when the Boundary is a Hyperplane.

In this section we use a freezing of the coefficients argument, as in Theorem **??**, to show that variable coefficient equations have $C^{2,\alpha}$ estimates up to the flat part of the boundary in an upper half ball.

Theorem 1. *Let $u \in C^{2,\alpha}(B^+_{2R}(0))$ be a solution, in $B^0_{2R}(0)$, to*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) \tag{3.1}$$

$$u(x) = 0 \text{ on } \{x_n = 0\} \cap B_{2R}(0) \tag{3.2}$$

Assume furthermore that $a_{ij}(x), b_i(x), c(x) \in C^\alpha(B^+_{2R})(0)$, that $a_{ij}(x) = a_{ji}(x)$, and that $a_{ij}(x)$ satisfy the following ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

for some constants $0 < \lambda \leq \Lambda$ and every $x \in B^+_{2R}(0)$ and all $\xi \in \mathbb{R}^n$.

Then there exists a constant $C = C(\lambda, \Lambda, n, a_{ij}, b_i, c)$ such that

$$\|u\|_{C^{2,\alpha}(B^+_R(0))} \leq$$

$$\leq C\left([f]_{C^\alpha(B^0_R(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

Proof: This proof mimics the proof of Theorem **??** (Theorem 1 in part 6). Therefore we will only indicate the minor differences. We may choose $\overline{B^+_{3R/2}(0)}$ as the compact set K and cover K by a finite number of balls $B_\delta(x^k)$ such that either $B_{4\delta}(x^k) \subset B^+_{3R/2}(0)$ or $B_{4\delta}(x^k) \cap B^+_{3R/2}(0) = B^+_{4\delta}(x^k)$. To estimate $\|D^2u\|_{C^\alpha_{\text{int},(2)}(x^k)}$ in the first case we may proceed exactly as in Theorem **??**. In case $B_{4\delta}(x^k) \cap B^+_{3R/2}(0) = B^+_{4\delta}(x^k)$ we may apply the boundary estimates from the previous section in place of Proposition **??** (Prop 1 in part 6). □

Corollary 3. *Let $u \in C^{2,\alpha}(B^+_{2R}(0))$ be a solution, in $B^0_{2R}(0)$, to*

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) \tag{3.3}$$

$$u(x) = g(x') \text{ on } \{x_n = 0\} \cap B_{2R}(0) \tag{3.4}$$

Assume furthermore that $f(x), a_{ij}(x), b_i(x), c(x) \in C^\alpha(B^+_{2R})(0)$, that $g \in C^{2,\alpha}(B'_{2R}(0))$, that $a_{ij}(x) = a_{ji}(x)$, and that $a_{ij}(x)$ satisfy the following ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

for some constants $0 < \lambda \leq \Lambda$ and every $x \in B^+_{2R}(0)$ and all $\xi \in \mathbb{R}^n$.

Then there exists a constant $C = C(\lambda, \Lambda, n, a_{ij}, b_i, c)$ such that

$$\|u\|_{C^{2,\alpha}(B^+_R(0))} \leq$$

$$\leq C\left(\|g\|_{C^{2,\alpha}(B'_{2R}(0))} + [f]_{C^\alpha(B^+_R(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

Proof: We may define $v(x) = u(x) - g(x')$. Then $Lv(x) = Lu(x) - Lg(x') = f(x) - Lg(x')$. We may thus define $\hat{f} = f(x) - Lg(x') \in C^\alpha(B^+_{2R}(0))$. Clearly

$$\|\hat{f}\|_{C^\alpha(B^+_{2R}(0))} \leq C\left(\|g\|_{C^{2,\alpha}(B'_{2R}(0))} + \|f\|_{C^\alpha(B^+_{2R}(0))}\right)$$

and

$$\|u\|_{C^{2,\alpha}(B^+_R(0))} \leq \|v\|_{C^{2,\alpha}(B^+_R(0))} + \|g\|_{C^{2,\alpha}(B^+_R(0))}. \tag{3.5}$$

We may apply the previous proposition on v with $\hat{f}$ in place of f and then estimate $\|u\|_{C^{2,\alpha}(B^+_R(0))}$ by (3.5). □

3.2 Boundary regularity for $C^{2,\alpha}$ boundaries.

Now we change our perspective to domains with boundaries that are locally given by the graph of a $C^{2,\alpha}$-function - which we will call $C^{2,\alpha}$-domains. The proofs are not that difficult since we may make a change of variables and transform the $C^{2,\alpha}-$domains to domains with the boundary given by a hyperplane and then use the estimates from the previous chapter.

We will use the notation $x' = (x_1, x_2, ..., x_{n-1})$ and $\nabla' = (\partial_1, \partial_2, ..., \partial_{n-1})$ etc. We will also always assume that the PDE we study satisfy the standard ellipticity condition:

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

for some constants $\Lambda, \lambda > 0$.

The next Lemma makes the important reduction of the $C^{2\alpha}-$domain to a domain that locally has the boundary contained in a hyperplane $\{x_n = 0\}$ which allows us to use the theory from the previous chapter. The method is commonly refereed to as a *"straightening of the boundary argument"*.

Lemma 2. *Let $g(x') \in C^{2,\alpha}(B'_{2R}(0))$, $g(0) = |\nabla' g(0)| = 0$ and*

$$\Omega = B_{2R}(0) \cap \{x_n > g(x')\}.$$

Assume furthermore that $u(x)$ is a solution in Ω to

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \qquad (3.6)$$

$$u(x) = 0 \text{ on } \{x_n = g(x')\} \cap B_{2R}(0) \qquad (3.7)$$

where L satisfies the assumptions of Theorem 1.

Then there exists a constant $c(\lambda, \Lambda) > 0$ such that if $|\nabla g(x')| \leq c(\lambda, \Lambda)$ then $v(x) = u(x_1, x_2, ..., x_{n-1}, x_n - g(x'))$ satisfies an elliptic equation in $\{(x', x_n - g(x')) \in \Omega\}$

$$\tilde{L}v(x) = \sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \tilde{b}_i(x)\frac{\partial v(x)}{\partial x_i} + \tilde{c}(x)v(x) = f(x', x_n - g(x')) \qquad (3.8)$$

$$v(x) = 0 \text{ on } \{x_n = 0\}, \qquad (3.9)$$

where $\tilde{a}_{ij}$, $\tilde{b}_i, \tilde{c} \in C^\alpha$ with $C^\alpha-$norms only depending on the corresponding norms for a_{ij}, b_i and c and the $C^{2,\alpha}-$norm of g. Furthermore, $\tilde{a}_{ij}$ satisfies the following ellipticity condition

$$\frac{\lambda}{2}|\xi|^2 \leq \sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\xi_i\xi_j \leq 2\Lambda|\xi|^2, \qquad (3.10)$$

for all $\xi \in \mathbb{R}^n$*.*

Proof: The proof is straight forward. We may write $u(x) = v(x', x_n + g(x'))$ and calculate

$$\frac{\partial u(x)}{\partial x_i} = \frac{\partial v(x', x_n + g(x'))}{\partial x_i} + \frac{\partial g(x')}{\partial x_i}\frac{\partial v(x', x_n + g(x'))}{\partial x_n} \text{ for } i \neq n,$$

and

$$\frac{\partial u(x)}{\partial x_n} = \frac{\partial v(x', x_n + g(x'))}{\partial x_n}.$$

Similarly we can express the second derivatives of u in terms of $v(x', x_n + g(x'))$ as follows, for $i, j \neq n$,

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_j} + \frac{\partial^2 g(x')}{\partial x_i \partial x_j}\frac{\partial v(x', x_n + g(x'))}{\partial x_n} +$$

$$+\frac{\partial g(x')}{\partial x_i}\frac{\partial g(x')}{\partial x_j}\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2} + \frac{\partial g(x')}{\partial x_i}\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_j \partial x_n},$$

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_n} = \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_n} + \frac{\partial g(x')}{\partial x_i}\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2} \text{ for } i \neq n,$$

and

$$\frac{\partial^2 u(x)}{\partial x_n^2} = \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2}.$$

In particular we have that

$$\sum_{i,j=1}^{n} \underbrace{a_{ij}(x)}_{=\tilde{a}_{ij}(x', x_n - g(x'))} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \sum_{i,j=1}^{n-1} a_{ij}(x)\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_j} +$$

$$+\sum_{i=1}^{n-1} \underbrace{\left(a_{in}(x) + \sum_{j=1}^{n-1} \frac{\partial g(x')}{\partial x_j} a_{ij} \right)}_{=\tilde{a}_{in}(x', x_n - g(x'))} \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_n}$$

$$+\underbrace{\left(a_{nn}(x) + 2\sum_{i=1}^{n-1} \frac{\partial g(x')}{\partial x_i} a_{in}(x) + \sum_{i,j=1}^{n-1} a_{ij}(x)\frac{\partial^2 g(x')}{\partial x_i \partial x_j} \right)}_{=\tilde{a}_{nn}(x', x_n - g(x'))} \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2} +$$

$$+\sum_{i,j=1}^{n-1} \frac{\partial^2 g(x')}{\partial x_i \partial x_j} a_{ij}(x)\frac{\partial v(x', x_n + g(x'))}{\partial x_n},$$

where the underbraces indicate how we define $\tilde{a}_{ij}$.

We can also calculate

$$\sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} = \sum_{i=1}^{n-1} b_i(x)\frac{\partial v(x', x_n + g(x'))}{\partial x_i} +$$

$$+\left(b_n(x) + \sum_{i=1}^{n-1} b_i(x)\frac{\partial g(x')}{\partial x_i}\right)\frac{\partial v(x', x_n + g(x'))}{\partial x_n}.$$

Setting $\tilde{b}_{x', x_n + g(x')} = b_i(x)$,

$$\tilde{b}_n(x', x_n + g(x')) = b_n(x) + \sum_{i=1}^{n-1} b_i(x)\frac{\partial g(x')}{\partial x_i} + \frac{\partial^2 g(x')}{\partial x_i \partial x_j} a_{ij}(x', x_n + g(x')),$$

and $\tilde{c}(x', x_n - g(x')))$ we see that

$$f(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) =$$

$$= \sum_{i,j=1}^{n} \tilde{a}_{ij}(x', x_n + g(x'))\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_j} +$$

$$+\sum_{i=1}^{n} \tilde{b}_i(x', x_n + g(x'))\frac{\partial v(x', x_n + g(x'))}{\partial x_i} + \tilde{c}(x', x_n + g(x'))v(x', x_n + g(x')).$$

Evaluating this at the point $(x', x_n - g(x'))$ gives

$$\tilde{L}v = \sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \tilde{b}_i(x)\frac{\partial v(x)}{\partial x_i} + \tilde{c}(x)v(x) = f(x', x_n - g(x')) = \tilde{f}(x),$$

where the last equality defines $\tilde{f}(x)$.

Next we show that $\tilde{L}$ is elliptic. Observe that if write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

and

$$G = \begin{bmatrix} 1 & 0 & 0 & \cdots & \frac{\partial g}{\partial x_1} \\ 0 & 1 & 0 & \cdots & \frac{\partial g}{\partial x_2} \\ 0 & 0 & 1 & \cdots & \frac{\partial g}{\partial x_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 + \frac{\partial g}{\partial x_n} \end{bmatrix}$$

then we may write the matrix $\tilde{A} = [\tilde{a}_{ij}]_{i,j=1}^n$ as follows

$$\tilde{A} = G^T AG.$$

It follows in particular that for any vector $\xi = (\xi_1, ..., \xi_n)^T \in \mathbb{R}^n$ that:

$$\sum_{i,j=1}^n \tilde{a}_{ij}\xi_i\xi_j = (G\xi)^T \cdot A \cdot (G\xi) \geq \lambda |G\xi|^2 \tag{3.11}$$

Clearly $|G\xi|^2 \geq \frac{1}{2}|\xi|$ if $|\nabla' g(x')| < c$ for some constant c. Notice that (3.11) states that

$$\sum_{i,j=1}^n \tilde{a}_{ij}\xi_i\xi_j \geq \lambda |G\xi|^2 \geq \frac{\lambda}{2}|\xi|^2,$$

if $|\nabla' g(x')| < c$, which is the left inequality in (3.10). The right inequality (3.10) in is proved in an analogous way.

To verify that $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}, \tilde{f} \in C^\alpha$ it is enough to verify that

$$\sup_{x,y} \frac{|a_{ij}(x', x_n + g(x')) - a_{ij}(y', y_n + g(y'))|}{|x-y|^\alpha} \text{ only depends on } \|g\|_{C^{2,\alpha}} \text{ and } \|a_{ij}\|_{C^\alpha} \tag{3.12}$$

and similarly for b_i, c_i and f. This since $\tilde{a}_{ij}$ is defined by terms $a_{ij}(x', x_n+g(x'))$ multiplied by derivatives of g - which are clearly in C^α. So if $a_{ij}(x', x_n+g(x')) \in C^\alpha$ then $\tilde{a}_{ij} \in C^\alpha$ by Proposition **??** (Prop 3 Part 6). To prove (3.12) we notice that

$$\sup_{x,y} \frac{|a_{ij}(x', x_n + g(x')) - a_{ij}(y', y_n + g(y'))|}{|x-y|^\alpha} =$$

$$= \sup_{x,y} \frac{|a_{ij}(x) - a_{ij}(y)|}{|(x', x_n - g(x')) - (y', y_n - g(y'))|^\alpha} \leq$$

$$\leq \left(\frac{|x-y|}{|(x', x_n - g(x')) - (y', y_n - g(y'))|} \right)^\alpha [a_{ij}]_{C^\alpha}.$$

But $\frac{|x-y|}{|(x', x_n - g(x')) - (y', y_n - g(y'))}| \leq C$ where C only depend on $\nabla' g$ and thus it follows that $\tilde{a}_{ij} \in C^\alpha$ with norm only depending on $\|f\|_{C^\alpha}$ and $\|g\|_{C^{2,\alpha}}$. □

Next we apply the straightening of the boundary argument to show regularity in $C^{2,\alpha}-$domains. We also allow non-zero boundary data.

Proposition 3. *Let $g(x') \in C^{2,\alpha}(B'_{2R}(0))$, $g(0) = |\nabla' g(0)| = 0$ and $|\nabla g(x')| \leq c$, where $c > 0$ is as in Lemma 2. Also let $\Omega = B_{2R}(0) \cap \{x_n > g(x')\}$.*

Then any solution $u(x)$ in Ω to the following PDF

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \tag{3.13}$$

$$u(x) = h(x) \text{ on } \{x_n = g(x')\} \cap B_{2R}(0), \tag{3.14}$$

where L satisfies the assumptions of Theorem 1, will satisfy the estimate

$$\|u\|_{C^{2,\alpha}(B_R(0)\cap\Omega)} \leq \tag{3.15}$$

$$\leq C\left(\|h\|_{C^{2,\alpha}(\partial\Omega)} + [f]_{C^\alpha(B_R^+(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B_{4R}^+(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B_{2R}^+(0))}\right),$$

where $C = C(\lambda, \Lambda, n, a_{ij}, b_i, c, g)$.

Proof: We define $v(x) = u(x', x_n - g(x'))$. Lemma 2 implies that v solves an elliptic equation

$$\tilde{L}v(x) = f(x', x_n - g(x')) \text{ in } \{(x', x_n - g(x')) \in \Omega\} \tag{3.16}$$

$$v(x) = h(x', x_n - gx')) \text{ on } \{x_n = 0\}, \tag{3.17}$$

where the coefficients of $\tilde{L}$ only depend on the coefficients of L and on g.

Corollary 3 implies that v satisfies the right estimates. But $u(x) = v(x', x_n + g(x')$ so a simple application of the chain rule for differentiation will imply that u satisfies (3.15). □

3.3 Global regularity

We are now ready to glue the boundary and the interior regularity together to prove global regularity. To that end we define $C^{2,\alpha}-$domains as domains whose boundaries can be covered by balls of some fixed radius such that the boundary can be represented by a $C^{2,\alpha}$ graph in each ball. See the figure below.

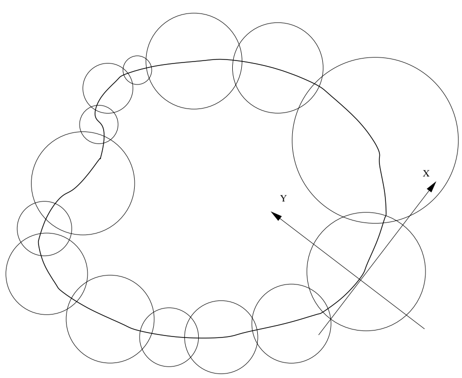

Figure 3.1: A $C^{2,\alpha}$-domain with the coordinate system for one ball $B_r(x^0)$ indicated.

Definition 1. *We say that a domain Ω is $C^{2,\alpha}$ if there exists an $r > 0$ such that for every $x^0 \in \partial\Omega$ there exists a coordinate system such that $B_r(x^0) \cap \partial\Omega$ is the graph of a $C^{2,\alpha}-$function in this coordinate system.*

With this definition at hand it is easy to prove $C^{2,\alpha}$ estimates for solutions to linear PDE. We may cover a neighborhood of the boundary by balls such that we can apply the boundary regularity in each ball. The rest of the domain can be covered by a compact set K with a mixed distance to the boundary. Using the interior regularity results we can estimate the $C^{2,\alpha}-$norm of the solution in the compact set.

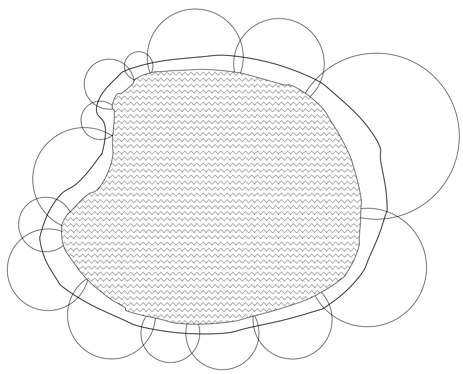

Figure 3.2: A $C^{2,\alpha}$-domain with a compact set (with the zig-zag pattern), where the solution is $C^{2,\alpha}$ by interior estimates, and a number of balls where the solution is $C^{2,\alpha}$ by the boundary estimates.

Theorem 2. *Assume that* $u \in C^2(\Omega)$, *where* Ω *is a bounded domain, is a solution to*

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) \text{ in } \Omega \tag{3.18}$$

$$u(x) = g(x) \text{ on } \partial\Omega. \tag{3.19}$$

Assume furthermore that $f(x), a_{ij}(x), b_i(x), c(x) \in C^\alpha(\Omega)$ *and that* $g(x) \in C^{2,\alpha}(\partial\Omega)$ *and that* Ω *is a* $C^{2,\alpha}-$*domain.*

Then there exists a constant C *such that*

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)} + \|u\|_{C(\Omega)}\right), \tag{3.20}$$

here $C = C(n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega)$ *where* $\Lambda, \lambda > 0$ *are the ellipticity constants of the PDE.*

Proof: We will prove the Theorem in three simple steps.

Step 1: *Cover the domain.*

Since Ω is a $C^{2,\alpha}-$domain we may cover the boundary $\partial\Omega$ by balls $B_{r/4}(z)$, $z \in \partial\Omega$, where $\Omega \cap B_r(x)$ is given by the graph of some function $g \in C^{2,\alpha}$. We may also decrease r, if necessary, to assure that $|\nabla g| < c$ in B_r' where c is as in Lemma 2.

Let $K = \{x \in \Omega;\ \text{dist}(x, \partial\Omega) \geq r/4\}$. Then K is compact and K together with the balls $B_{r/2}(z)$, $z \in \partial\Omega$, will cover Ω.

Step 2: *Local bounds for the norm.*

For any ball $y \in \Omega$ we will either have $B_{r/4}(y) \subset K$ or $B_{r/4}(y) \subset B_{r/2}(z)$ for some $z \in \partial\Omega$.

If $B_{r/y}(y) \subset K$ then Theorem **??** will imply that

$$\|u\|_{C^{2,\alpha}(B_{r/4}(y))} \leq \|u\|_{C^{2,\alpha}(K)} \leq \frac{C}{r^{2+\alpha}} \left(\|f\|_{C^\alpha(\Omega)} + \|u\|_{C(\Omega)}\right), \tag{3.21}$$

where we have used that $\text{dist}(K, \partial\Omega) = r/4$ and thus $\text{dist}(K, \partial\Omega)^{-(2+\alpha)} = 2^{-(2+\alpha)} r^{-(2+\alpha)}$ and that the factor $2^{-(2+\alpha)}$ may be included in the constant C.

And if $B_{r/4}(y) \subset B_{r/2}(z)$ then Proposition 3 will imply that

$$\|u\|_{C^{2,\alpha}(B_{r/4}(y))\cap\Omega} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right). \tag{3.22}$$

Step 3: *Global estimates and the conclusion of the Theorem.*

Clearly (3.21) and (3.22) together implies that for any $x \in \Omega$

$$|\nabla u(x)| + |D^2 u(x)| \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right), \tag{3.23}$$

where $C = C(n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega)$ where we included the r dependence in the dependence on Ω.

Therefore we only need to show that

$$\frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right). \tag{3.24}$$

We will consider two (or three - depending on how you count) cases. Either $|x-y| < r/4$ or $|x-y| \geq r/4$. If $|x-y| < r/4$ and both $x, y \in K$ then (3.24) follows from (3.21) and if one of x or y, lets say y for definiteness, satisfies $y \notin K$ then there must exists a ball $B_{r/4}(z)$, $z \in \partial\Omega$ such that $y \in B_{r/4}(z)$. But then, since $|x-y| < r/4$, both $x, y \in B_{r/4}(y) \subset B_{r/2}(z)$ and (3.24) follows from (3.22). In any case, (3.24) follows if $|x-y| < r/4$.

If $|x-y| \geq r/4$ then

$$\frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \leq \frac{4^\alpha}{r^\alpha}|D^2 u(x) - D^2 u(y)| \leq 2\frac{4^\alpha}{r^\alpha}\sup_{x\in\Omega}|D^2 u(x)| \leq \tag{3.25}$$

$$\leq 2\frac{24^\alpha C}{r^\alpha}\left(\|f\|_{C^\alpha(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right),$$

where we used (3.23) in the last inequality. Notice that the constant in the right hand side of (3.25) only depend on r and $n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega$. But r only depend on Ω so we may conclude that the constant in (3.25) will only depend on $n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c$ and Ω.

We have thus proved (3.24) which together with (3.23) implies the estimate (3.20). $\square$