Homework Assignment 2, Selected topics in PDE, Fall 2014

Name:______________________________________ **Personal Id Number:**______________________

email:____

We have shown, during the lectures, that bounded solutions to elliptic PDEs

$$\begin{array}{ll} Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) & \text{in } B_2^+(0) \\ u(x) = g(x') & \text{on } \{x_n = 0\} \cap B_2(0) \end{array} \tag{1}$$

are in $C^{2,\alpha}(B_1^+(0))$ if $f, a_{ij} \in C^\alpha$ and $g \in C^{2,\alpha}$.

It is also important to get estimates on u if $g(x')$ has worse regularity than $C^{2,\alpha}$. This is a vast field of inquiry but in this assignment we will take a first step in that direction.

Your assignment is to prove the following Theorem.

Theorem 1. *Let $u(x) \in C^2_{loc}(B_2^+(0))$ be a solution to (1). Furthermore assume that $a_{ij}(x), f(x) \in C^\alpha(B_2^+(0))$ and that $g(x') \in C^\alpha(\mathbb{R}^{n-1})$ for some $\alpha \in (0,1)$. Then*

$$\|u\|_{C(B_1^+)} + \|\nabla u\|_{C_{int,(1-\alpha)}(B_1^+)} + \|D^2 u\|_{C_{int,(2-\alpha)}(B_1^+)} + [D^2 u]_{C^\alpha_{int,(2-\alpha)}(B_1^+)} \le C\left(\|f\|_{C^\alpha(B_2^+(0))} + \|g\|_{C^\alpha(B_2'(0))} + \|u\|_{C(B_2^+(0))}\right),$$

where $C = C(n, \alpha, \lambda, \Lambda, a_{ij})$.

Furthermore

$$\|u\|_{C^\alpha(B_1^+(0))} \le C\left(\|f\|_{C^\alpha(B_2^+(0))} + \|g\|_{C^\alpha(B_2'(0))} + \|u\|_{C(B_2^+(0))}\right),$$

where $C = C(n, \alpha, \lambda, \Lambda, a_{ij})$.

You may prove the theorem in any way you want but you might want to use the following steps.

1. Define the $g_\epsilon(x') = \int_{\mathbb{R}^{n-1}} g(y')\phi_\epsilon(x'-y')dy'$ where ϕ_ϵ is the standard mollifier on $\mathbb{R}^{n-1}$. Prove that $g(x') - C\epsilon^\alpha[g]_{C^\alpha(\mathbb{R}^{n-1})} \le g_\epsilon(x') \le g(x') + C\epsilon^\alpha[g]_{C^\alpha(\mathbb{R}^{n-1})}$ where C only depend on the mollifier ϕ.
2. Prove that $[D^3 g_\epsilon(x')]_{C(\mathbb{R}^{n-1})} \le \frac{C}{\epsilon^{3-\alpha}}[g]_{C^\alpha(\mathbb{R}^{n-1})}$.

   Hint: *Notice that $\int_{\mathbb{R}^{n-1}} \frac{\partial^3 \phi_\epsilon(x'-y')}{\partial x_i \partial x_j \partial x_k} g(x')dy' = 0$ and thus $\int_{\mathbb{R}^{n-1}} \frac{\partial^3 \phi_\epsilon(x'-y')}{\partial x_i \partial x_j \partial x_k} g(y')dy' = \int_{\mathbb{R}^{n-1}} \frac{\partial^3 \phi_\epsilon(x'-y')}{\partial x_i \partial x_j \partial x_k}\left(g(y') - g(x')\right)dy'$*
3. Use the mean value property to show that $[D^2 g_\epsilon]_{C^\alpha(\mathbb{R}^{n-1})} \le \frac{C}{\epsilon^2}[g]_{C^\alpha(\mathbb{R}^{n-1})}$.
4. Prove that if $u_\epsilon^\pm$ solves (1)[1] with boundary data $g_\epsilon(x') \pm C\epsilon^\alpha[g]_{C^\alpha(\mathbb{R}^{n-1})}$ then $u_\epsilon^-(x) \le u(x) \le u^+(x)$, in particular $0 \le u(x) - u_\epsilon^-(x) \le C\epsilon^\alpha[g]_{C^\alpha(\mathbb{R}^{n-1})}$.
5. Fix a ball $B_\epsilon(y)$ where $2\epsilon = y_n$ and write $u(x) = u_\epsilon^-(x) + w(x)$ and estimate ∇u, $D^2 u(x)$ and $\frac{|D^2(u)(x^1) - D^2 u(x^2)|}{|x^1 - x^2|^\alpha}$ in $B_\delta(y)$ by using estimates for u_ϵ^- and w.
6. Tie the above steps together to prove the first part of the Theorem.
7. Prove the second part of the theorem. You may want to use the following steps:
   (a) Let $x^1, x^2 \in B_1^+(0)$ and assume, without loss of generality that $x_n^1 \le x_n^2$.
   (b) If $|x^1 - x^2| \le x_n^1$ prove that $\frac{|u(x^1) - u(x^2)|}{|x^1 - x^2|^\alpha}$ is bounded by using the mean value theorem from calculus.
   (c) If $|x^1 - x^2| < x_n^1$ write

   $$|u(x^1) - u(x^2))| \le \left|u(x^1) - g((x^1)', 0)\right| + \left|g((x^1)', 0) - g((x^2)', 0)\right| + \left|g((x^2)', 0) - u(x^2)\right|$$

   and estimate the three terms to the right.

I have no idea whether this assignment is easy or difficult. The point is that you should engage the text and get some non-trivial questions to think about. If it turns out that the assignment is very difficult please do not hesitate to contact me with any questions.

Your solutions, together with this sheet, should be handed in (preferably) during the lecture on the 16th December. The final deadline is Friday the 19th December.

Good Luck!

[1] You don't have to prove the existence of this solution.