SF2729 Groups and Rings
Problem set 7

due: Friday January 30 in class.

Write clear, clean, brief, and complete solutions and use whole sentences. Solutions without proper reasoning score worse. You can submit hand-written or typed solutions and turn them in in class or send them by email to `wojtek@kth.se`. I will not accept late homework except under extraordinary circumstances that you need to discuss with me before the deadline.

Problem 1. On the set $P(X)$ of subsets of a set X, define an addition by $S + T = (S \cup T) - (S \cap T)$ and a multiplication by $S \times T = S \cap T$. Show that with this structure, $P(X)$ is a commutative ring with identity. Show also that all elements different from 1 are zero-divisors.

Problem 2. Let $S = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$ be the ring of integral Hamilton Quaternions. Define $N\colon S \to \mathbb{Z}$ to be the following function:

$$N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$$

Show that $N(\alpha\beta) = N(\alpha)N(\beta)$. Use it to prove that α in S is invertible if and only if $N(\alpha) = 1$. Conclude that the group of units $S^\times$ is isomorphic to the Quaternion group of order 8.

Problem 3. Let R be a commutative ring $p(X)$ be a polynomial in $R[X]$. Show that $p(X)$ is a zero-divisor if and only there is a nonzero element b in R, such that $bp(X) = 0$.