Matematiska Institutionen
KTH

Exam to the course Discrete Mathematics, SF2736, January 14, 2015, 14.00-19.00.

Observe:

1. Nothing else than pencils, rubber, rulers and papers may be used.
2. Bonus marks from the homeworks will be added to the sum of marks on part I. The maximum number of marks on part I is 15.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.
4. **Observe.** All answers must be justified with a complete argumentation!!

Part I

1. (3p) Find the number of ways to distribute eleven identical yellow balloons and eight identical red balloons to five children. The answer must be given as an integer, or as a product of integers.
2. (a) (1.5p) Find the greatest common divisor of the integers 518, 434 and 732.

   (b) (1.5p) Find the least common multiple of the integers 518, 434 and 732.
3. (3p) Find and describe a 4-regular graph $\mathcal{G}$ without multiple edges and loops such that $\mathcal{G}$ admits an Euler circuit but no Hamilton cycle.

   (4-regular means that every vertex has degree 4. A loop is an edge the endpoints of which is the same vertex.)
4. Let $\mathcal{S}_6$ denote the group of all permutations of the elements in the set $\{1, 2, \ldots, 6\}$. Let $\varphi = (1\ \ 6\ \ 2\ \ 5)(4\ \ 5\ \ 2)$ and $\psi = (1\ \ 5)(2\ \ 3)$ be two elements of $\mathcal{S}_6$.

   (a) (1p) Find a permutation γ in $\mathcal{S}_6$ such that $\varphi^2\gamma = \psi$.

   (b) (1p) Find two distinct permutations δ in $\mathcal{S}_6$ such that $\delta^2 = \psi$.

   (c) (1p) Is there any permutation β in $\mathcal{S}_6$ such that $\beta^2 = \varphi$.
5. (3p) Consider a regular polygon with 12 unlabeled edges. (The polygon can be rotated and flipped.) Find the number of ways to color the edges in q distinct colors.

Part II

6. The group $G = (Z_{23} \setminus \{0\}, \cdot)$ is a cyclic group.

   (a) (1p) Show that the element 5 generates G.

   (b) (2p) Find another element of G that generates G.

7. (4p) Find the number of equivalence relations $\mathcal{R}$ on a set $M = \{1, \dots, 9\}$ such that $|\mathcal{R}| = 27$ and

$$|\{x \in M \mid 1\mathcal{R}x\}| = |\{x \in M \mid 2\mathcal{R}x\}| = |\{x \in M \mid 3\mathcal{R}x\}| = 3.$$

8. (4p) Let G be a graph with vertex set V and edge set E, with no multiple edges and no loops (a loop is an edge the endpoints of which is one single vertex). Assume that the lengths of the cycles belong to the set $\{8, 10, 12, 14\}$ and that

$$\{\delta(v) \mid v \in V\} = \{9, 11, 13, 15, 17\},$$

where $\delta(v)$ denotes the valency (degree) of the vertex v. Show that, for every integer q in the interval $19 \leq q \leq 72$, the edges can be colored in exactly q distinct colors such that no two edges of the same color meet at a vertex. (The number of possibilities for q can certainly be proved to be larger than this interval, but to get 4p it is enough to verify the given interval of possibilities.)

Part III

9. Let as usual $\mathcal{S}_n$ denote the group consisting of all permutations of the elements in the set $\{1, 2, \dots, n\}$.

   (a) (1p) Does an equation $\psi^2 = \varphi$ have a solution ψ for any even permutation φ.

   (b) (1p) Derive a formula for the number of solutions ψ in $\mathcal{S}_n$ to the equation $\psi^4 = \text{Id.}$?

   (c) (3p) Can the number of solutions ψ in $\mathcal{S}_n$ to an equation $\psi^2 = \varphi$, where $\varphi \neq \text{Id.}$, be larger than the number of solutions in $\mathcal{S}_n$ to the equation $\psi^2 = \text{Id.}$.

10. (5p) Evaluate the new idea below for the construction of an 1-error-correcting binary code C. Discuss whether the construction is fruitful, give its advantages and disadvantages in comparison with the traditional Hamming construction of 1-error-correcting codes.

    Idea. Use a binary $k \times n$-matrix $\mathbf{H}$ and two distinct $k \times 1$-matrices $\mathbf{b}_1$ and $\mathbf{b}_2$ to define a code C by

    $$C = \{\mathbf{c} = (c_1 \ \dots \ c_n) \mid \mathbf{H}\mathbf{c}^T = \mathbf{b}_1\} \cup \{\mathbf{c} = (c_1 \ \dots \ c_n) \mid \mathbf{H}\mathbf{c}^T = \mathbf{b}_2\}.$$

    Also discuss further possible generalizations of this construction.