Matematiska Institutionen
KTH

Solution of problem 8b of a typical exam to the course Discrete Mathematics, SF2736, fall 14.

8. (3p) Consider any group G. Is it true in general that if $a+H$ and $b+K$ are cosets to subgroups H and K of G then the number of elements in $(a+H)\cap(b+K)$ either are equal to zero or divides the number of element of G, i.e., is it in general true that

$$|(a+H)\cap(b+K)| = m \quad \text{and} \quad |G| = n \quad \Longrightarrow \quad m = 0 \quad \text{or} \quad m \mid n.$$

Solution. Yes it is true. To prove that, we first note that $H\cap K$ is a subgroup of H, which is a subgroup of G. Thus, by the theorem of Lagrange, $|H\cap G|$ divides $|H|$, that divides $|G|$. It thus suffices to show that

$$|(a+H)\cap(b+K)| = |H\cap K|,$$

if $(a+H)\cap(b+K)\neq\emptyset$
Assume that $x\in(a+H)\cap(b+K)$. Then, for some $h\in H$ and $k\in K$

$$x = a+h = b+k,$$

and, for every $g\in H\cap K$,

$$x+g = a+h+g = b+k+g = a+h' = b+k' \in (a+H)\cap(b+K),$$

(where $h' = h+g\in H$ and $k' = k+g\in K$). This shows that

$$x+(H\cap K)\subseteq(a+H)\cap(b+K).$$

Assume that $x'' = a+h'' = b+k''$ is an element of $(a+H)\cap(b+K)$. Then

$$x''-x = h''-h = k''-k = g''\in H\cap K,$$

which shows that $x'' = x+g''\in x+(H\cap K)$. Hence,

$$(a+H)\cap(b+K)\subseteq x+(H\cap K).$$

We have thus verified that

$$(a+H)\cap(b+K) = x+(H\cap K),$$

i.e., that $(a+H)\cap(b+K)$ is a coset to $H\cap K$ and thus of the same size as $H\cap K$.