Matematiska Institutionen
KTH

Solutions to homework number 2 to SF2736, fall 2014.

Please, deliver this homework at latest on Monday, November 24, 2014. Provide both your name and your e-mail address with your solutions.

The homework must be delivered individually, and, in general, just hand-written notes are accepted. You are allowed to discuss the problems with your classmates, but you are not allowed to deliver a copy of the solution of another student.

1. (0.1p) Let $M = \{1, 2, \ldots, 9\}$. Find the smallest equivalence relation $\mathcal{R}$ such that $\mathcal{R}$ contains the set

$$\mathcal{A} = \{(1, 2), (5, 2), (3, 3), (4, 9), (5, 7), (7, 7), (9, 3)\}.$$

Solution. Every equivalence relation is in $1 - 1$-correspondence to a partition of the set M into mutually disjoint subsets. Elements are related to their set-mates, but to no other elements. The equivalence relation with fewest number of relations is then given by the partition

$$\{1, 2, \ldots, 9\} = \{1, 2, 5, 7\} \cup \{3, 4, 9\} \cup \{6\} \cup \{8\}.$$

2. (0.1p) Let M, $\mathcal{A}$ and $\mathcal{R}$ be as above, Is there any subset $\mathcal{B}$ of $M \times M$ such that $|\mathcal{B}| < |\mathcal{A}|$ and such that $\mathcal{R}$ is the smallest equivalence relation containing $\mathcal{B}$.

Solution. Pairs of type (a, a) can be deleted, so

$$\mathcal{B} = \{(1, 2), (5, 2), (3, 3), (4, 9), (5, 7), (7, 7), (9, 3)\} \setminus \{(3, 3)\}$$

and several other subsets will do.

3. (0.2p) Is there any equivalence relation $\mathcal{R}$ of size 21 (i.e., $|\mathcal{R}| = 21$) on some set M such that $\mathcal{R}$ induces a partition of M into four equivalence classes.

Solution. Yes, every equivalence class of size r contributes with r^2 pairs (x, y) to the equivalence relation. We note that

$$21 = 3^2 + 1^2 + 1^2 + 1^2,$$

so the following partition of the set $\{1, 2, 3, 4, 5, 6\}$ into equivalence classes:

$$\{1, 2, 3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\} \cup \{4\} \cup \{5\} \cup \{6\}$$

has the desired property.

4. (0.1p) Find countable infinite sets A and B and a function f from A to B such that if a function g from B to A is such that $g \circ f$ is the identity on A then $f \circ g$ must be the identity on B.

   Solution. Let $A = B = \mathbb{N}$, that is, the set of positive integers. Let $f(n) = n$, for $n \in \mathbb{N}$. If $(g \circ f)(n) = n$, then $g(n) = n$. Thus

   $$(f \circ g)(n) = f(g(n)) = f(n) = n,$$

   and consequently, $f \circ g$ is the identity on B.

5. Let A and B be countable infinite sets. Are there any functions f from A to B such that the number of functions g from B to A with the property that $g \circ f$ is the identity on A, while $f \circ g$ is not the identity on B, is

   (a) (0.1p) finite,

   Solution. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$, and let $f(a_i) = b_i$, for $i = 1, 2, \ldots$. If f is surjective then there is just one map g with the property $g \circ f$ is the identity, and for such a map $f \circ g$ is the identity on B, compare the previous problem for an explanation of this fact. If f is not surjective, then we can always find an infinite set of functions g such that $f \circ g$ is the identity on B, see the solution of the problem below.
   ANSWER: No.

   (b) (0.2p) countable infinite,

   Solution. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$, and let $f(a_i) = b_{i+1}$, for $i = 1, 2, \ldots$. If $g \circ f$ is the identity on A, then necessarily $g(b_i) = a_{i-1}$, for $i = 2, 3, \ldots$. For $g(b_1)$ we can choose any element in the set A. Thus,
   ANSWER: Yes.

   (c) (0.2p) uncountable infinite.

   Solution. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$, and let $f(a_i) = b_{2i-1}$, for $i = 1, 2, \ldots$. Define g_α, where α is a real number with decimal expansion

   $$\alpha = 0.r_1 r_2 r_3 \ldots \qquad r_i \in \{0, 1, 2, \ldots, 9\},$$

   by

   $$g_\alpha(n) = \begin{cases} a_i & \text{if} \quad n = 2i - 1, \\ a_{r_i} & \text{if} \quad n = 2i. \end{cases}$$

   Then $g_\alpha \circ f$ is the identity on A, while $f \circ g_\alpha$ is not the identity on B. Hence, the set of functions g such that $f \circ g$ is not the identity on B contains a subset which is uncountable infinite, and thus itself is uncountable infinite.
   ANSWER: Yes.