Matematiska Institutionen
KTH

Solutions to homework number 3 to SF2736, fall 2014.

Please, deliver this homework at latest on Monday, December 1, 2014. Provide both your name and your e-mail address with your solutions.

The homework must be delivered individually, and, in general, just handwritten notes are accepted. You are allowed to discuss the problems with your classmates, but you are not allowed to deliver a copy of the solution of another student.

1. (0.1p) Find the integer that is equal to

$$\sum_{i=1}^{7} 2^i \binom{7}{i}.$$

Solution. From the binomial theorem we get that

$$(1+2)^7 = \sum_{i=0}^{7} 2^i \binom{7}{i}.$$

Thus,

$$\sum_{i=1}^{7} 2^i \binom{7}{i} = (1+2)^7 - 2^0 \binom{7}{0} = 3^7 - 1 = 2187 - 1.$$

ANSWER: 2186.

2. (0.2p) How many "words" can you form with the letters in the word MISSISSIPPI such that no four S are adjacent, no for I are adjacent and no two P are adjacent. For instance the word IPPISSISSIM is an allowed word while the words MIIIISSPSSP and MIII are not accepted.

Solution. We use the principle of inclusion-exclusion. Let A denote the set of words with 4 consecutive I, B the set of words with 4 consecutive S and C the set of words with 2 consecutive P.

There are 11 possible position for the letters, thus the total number of words is

$$\binom{11}{4,4,2,1}.$$

Sticking the I's, the S's and the P's together to one large letter, respectively, reduces the number of letters to use in order to form words. We then get

$$|A| = \binom{8}{1,4,2,1}, \qquad |A| = \binom{8}{4,1,2,1}, \qquad |A| = \binom{10}{4,4,1,1},$$

$$|A \cap B| = \binom{5}{1,1,2,1}, \quad |A \cap C| = \binom{7}{1,4,1,1}, \quad |C \cap B| = \binom{7}{4,1,1,1},$$

and

$$|A \cap B \cap C| = \binom{4}{1,1,1,1},$$

Thus

ANSWER:

$$\binom{11}{4,4,2,1} - 2\binom{8}{1,4,2,1} - \binom{10}{4,4,1,1} + 2\binom{7}{1,4,1,1} + \binom{5}{1,1,2,1} - \binom{4}{1,1,1,1}.$$

3. (0.3p) Find all integers n and k such that

$$\binom{n}{k} = 517.$$

Solution. We will use the fact that

$$0 \leq k < k' \leq \frac{n}{2} \qquad \Longrightarrow \qquad \binom{n}{k} < \binom{n}{k'} \tag{1}$$

Now

$$517 = 47 \cdot 11.$$

The given equation is equivalent to the equation

$$n(n-1)\cdots(n-k+1) = k! \cdot 47 \cdot 11.$$

As 47 is a prime number we get that 47 divides $(n-i)$ for some $0 \leq i \leq k-1$.

If $k = 2$ then we would have

$$\binom{n}{2} = \frac{n(n-1)}{2} \geq \frac{47 \cdot 46}{2} = 1058 > 517.$$

Now, using Equation (1) and the well-known fact that

$$\binom{n}{k} = \binom{n}{n-k},$$

we may conclude that

$$\binom{n}{k} \geq 1058$$

for $2 \leq k \leq n-2$. We can thus conclude that the only remaining possibilities to check is if

$$\binom{n}{1} = 517 \qquad \text{or} \qquad \binom{n}{n-1} = 517,$$

for some n. Trivially, this is true in the case $n = 517$. Hence

ANSWER: $n = 517$ and k is either 1 or 516.

4. (0.4p) A class has 7 boys and 8 girls. Find the number of ways to partition the class into four (mutually disjoint) unlabeled subgroups such that each subgroup contains at least one girl and such that the girls A and B will not belong to the same group. Answer with an integer (and motivations).

Solution. The total number of partitions of the girls into four non-empty subgroups is equal to the Stirling number $\mathrm{S}(8,4)$. The number of ways when girl A and girl B are in the same group is equal to the Stirling number $\mathrm{S}(7,4)$, as we in that case can regard A and B as one individual. The number of ways to form partitions of the set of girls as specified is thus

$$\mathrm{S}(8,4) - \mathrm{S}(7,4).$$

For each such partition it remains to distribute the boys. Each boy have four choices of groups. Thus, (and by using the recursion

$$\mathrm{S}(n,k) = \mathrm{S}(n-1,k-1) + k \cdot \mathrm{S}(n-1,k),$$

to calculate the Stirling numbers), we get

ANSWER:

$$(\mathrm{S}(8,4) - \mathrm{S}(7,4)) \cdot 4^7 = (1701 - 350) \cdot 16384 = 22134784.$$