Matematiska Institutionen
KTH

Solutions to homework number 4 to SF2736, fall 2014.

Please, deliver this homework at latest on Monday, December 8, 2014. Provide both your name and your e-mail address with your solutions.

The homework must be delivered individually, and, in general, just hand-written notes are accepted. You are allowed to discuss the problems with your classmates, but you are not allowed to deliver a copy of the solution of another student.

1. (0.2p) Consider the symmetrical group $\mathcal{S}_{10}$ of permutations of the elements in the set $\{1, 2, \ldots, 10\}$. For which integers k are there elements φ in $\mathcal{S}_{10}$ of order k.

   Solution. The order of a permutation φ is the least common multiple of the lengths of the cycles when φ is written as a product of disjoint cycles, the sum of the lengths of the cycles add to 10. We get the following partitions of 10 into a sum of two integers

   $$10 = 10 + 0 = 9 + 1 = 8 + 2 = 7 + 3 = 6 + 4 = 5 + 5.$$

   Each term can then be split further, e.g., $10 = 5 + 3 + 2 = 5 + 1 + 4$. We thus get

   ANSWER: 10, 9, 8, 21, 12, 5, 14, 15, 7, 6, 10, 20, 30, 4, 3, 2, 1.

2. (0.2p) Let φ and ψ denote the following elements in the symmetrical group $\mathcal{S}_6$:

   $$\varphi = (1\ \ 2\ \ 4)(3\ \ 5)(2\ \ 6), \qquad \psi = (1\ \ 5\ \ 3\ \ 4)(3\ \ 5\ \ 2)$$

   Is there any element γ in $\mathcal{S}_6$ such that

   $$\varphi\gamma\varphi\gamma\varphi = \psi.$$

   Solution. The permutations φ is an even permutation and ψ is an odd permutation, as they are products of an even and odd, respectively, number of 2-cycles:

   $$\varphi = (1\ \ 2)(1\ \ 4)(3\ \ 5)(2\ \ 6), \qquad \psi = (1\ \ 4)(1\ \ 3)(1\ \ 5)(3\ \ 2)(3\ \ 2).$$

   As a product of two odd permutation is an even permutation, a product of an even and an odd permutation is an odd permutation we get that the left side of the given equality, for every permutation γ, is an even permutation. The right side is an odd permutation. Hence, no permutation γ can be found satisfying the equatio.

3. (0.3p) Find two non-Abelian groups of size 12 that are not isomorphic.

   Solution. We consider two subgroups of $\mathcal{S}_6$, one is the dihedral group D_6 consisting of rotations and reflections of the regular hexagon with vertices $\{1, \ldots, 6\}$, containing the elements

   $$D_6 = \{\varphi = (1\ \ 2\ \ ...\ \ 6), \varphi^2, \ldots ..., \varphi^6, (1\,6)(2\,5)(3\,4), \ldots, (1)(2\,6)(3\,5)(4)\}.$$

   The other subgroup consists of all even permutations on the subset $\{1, \ldots, 4\}$ of $\{1, \ldots, 6\}$. The latter has no element of order 6, while the first group has. Thus they cannot be isomorphic.

4. (0.3p) The set $U(\mathbb{Z}_{24})$ of multiplicatively invertible elements in $\mathbb{Z}_{24}$ constitutes an Abelian group. This group is isomorphic to a direct product S of cyclic groups. Find this direct product S.

   Solution. We start by finding the order of the elements in the group

   $$G = U(\mathbb{Z}_{24}) = \{1, 5, 7, 11, 13, 17, 19, 23\}$$

   There orders are

   $$o(5) = 2,\ o(7) = 2,\ o(11) = 2,\ o(13) = 2,\ o(17) = 2,\ o(19) = 2,\ o(23) = 2$$

   If G is a direct product of cyclic groups $C_i = \langle g_i \rangle$ with i elements, for some integers i

   $$G = C_{i_1} \times \cdots \times C_{i_t}$$

   then G must contain elements of order i_1, ..., i_t, respectively, the elements

   $$(g_{i_1}, e, \ldots, e), \qquad \ldots, \qquad (e, \ldots, e, g_{i_t}).$$

   Thus, regarding the order of the elements $U(\mathbb{Z}_{24})$, the only possibility is that G is isomorphic with a direct product of groups with exactly two elements. As the size of G is 8 we thus get

   ANSWER: $(\mathbb{Z}_2, +) \times (\mathbb{Z}_2, +) \times (\mathbb{Z}_2, +)$.