distance between them. The following example illustrates the latter problem.

EXAMPLE 9 Distance Between Parallel Planes

The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $(1, 2, -2)$ and $(2, 4, -4)$, are parallel vectors. Find the distance between these planes.

Solution

To find the distance D between the planes, we may select an arbitrary point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. From (9), the distance between P_0 and the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|2(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6} \quad \blacksquare$$

1. Find a point-normal form of the equation of the plane passing through P and having $\mathbf{n}$ as a normal.
   (a) $P(-1, 3, -2)$; $\mathbf{n} = (-2, 1, -1)$ (b) $P(1, 1, 4)$; $\mathbf{n} = (1, 9, 8)$
   (c) $P(2, 0, 0)$; $\mathbf{n} = (0, 0, 2)$ (d) $P(0, 0, 0)$; $\mathbf{n} = (1, 2, 3)$
2. Write the equations of the planes in Exercise 1 in general form.
3. Find a point-normal form of the equations of the following planes.
   (a) $-3x + 7y + 2z = 10$ (b) $x - 4z = 0$
4. Find an equation for the plane passing through the given points.
   (a) $P(-4, -1, -1)$, $Q(-2, 0, 1)$, $R(-1, -2, -3)$
   (b) $P(5, 4, 3)$, $Q(4, 3, 1)$, $R(1, 5, 4)$
5. Determine whether the planes are parallel.
   (a) $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$
   (b) $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$
   (c) $2y = 8x - 4z + 5$ and $x = \frac{1}{2}z + \frac{1}{4}y$
6. Determine whether the line and plane are parallel.
   (a) $x = -5 - 4t,\ y = 1 - t,\ z = 3 + 2t$; $x + 2y + 3z - 9 = 0$
   (b) $x = 3t,\ y = 1 + 2t,\ z = 2 - t$; $4x - y + 2z = 1$
7. Determine whether the planes are perpendicular.
   (a) $3x - y + z - 4 = 0$, $x + 2z = -1$ (b) $x - 2y + 3z = 4$, $-2x + 5y + 4z = -1$
8. Determine whether the line and plane are perpendicular.
   (a) $x = -2 - 4t,\ y = 3 - 2t,\ z = 1 + 2t$; $2x + y - z = 5$
   (b) $x = 2 + t,\ y = 1 - t,\ z = 5 + 3t$; $6x + 6y - 7 = 0$
9. Find parametric equations for the line passing through P and parallel to $\mathbf{n}$.
   (a) $P(3, -1, 2)$; $\mathbf{n} = (2, 1, 3)$ (b) $P(-2, 3, -3)$; $\mathbf{n} = (6, -6, -2)$
   (c) $P(2, 2, 6)$; $\mathbf{n} = (0, 1, 0)$ (d) $P(0, 0, 0)$; $\mathbf{n} = (1, -2, 3)$
11. Find parametric equations for the line of intersection of the given planes.
   (a) $7x - 2y + 3z = -2$ and $-3x + y + 2z + 5 = 0$
   (b) $2x + 3y - 5z = 0$ and $y = 0$
12. Find the vector form of the equation of the plane that passes through P_0 and has normal $\mathbf{n}$.
   (a) $P_0(-1, 2, 4)$; $\mathbf{n} = (-2, 4, 1)$ (b) $P_0(2, 0, -5)$; $\mathbf{n} = (-1, 4, 3)$
   (c) $P_0(5, -2, 1)$; $\mathbf{n} = (-1, 0, 0)$ (d) $P_0(0, 0, 0)$; $\mathbf{n} = (a, b, c)$
13. Determine whether the planes are parallel.
   (a) $(-1, 2, 4) \cdot (x - 5, y + 3, z - 7) = 0$; $(2, -4, -8) \cdot (x + 3, y + 5, z - 9) = 0$
   (b) $(3, 0, -1) \cdot (x + 1, y - 2, z - 3) = 0$; $(-1, 0, 3) \cdot (x + 1, y - z, z - 3) = 0$
14. Determine whether the planes are perpendicular.
   (a) $(-2, 1, 4) \cdot (x - 1, y, z + 3) = 0$; $(1, -2, 1) \cdot (x + 3, y - 5, z) = 0$
   (b) $(3, 0, -2) \cdot (x + 4, y - 7, z + 1) = 0$; $(1, 1, 1) \cdot (x, y, z) = 0$
15. Find the vector form of the equation of the line through P_0 and parallel to $\mathbf{v}$.
   (a) $P_0(-1, 2, 3)$; $\mathbf{v} = (7, -1, 5)$ (b) $P_0(2, 0, -1)$; $\mathbf{v} = (1, 1, 1)$
   (c) $P_0(2, -4, 1)$; $\mathbf{v} = (0, 0, -2)$ (d) $P_0(0, 0, 0)$; $\mathbf{v} = (a, b, c)$
16. Show that the line

$$x = 0, \quad y = t, \quad z = t \quad (-\infty < t < +\infty)$$

   (a) lies in the plane $6x + 4y - 4z = 0$
   (b) is parallel to and below the plane $5x - 3y + 3z = 1$
   (c) is parallel to and above the plane $6x + 2y - 2z = 3$
17. Find an equation for the plane through $(-2, 1, 7)$ that is perpendicular to the line $x - 4 = 2t$, $y + 2 = 3t$, $z = -5t$.
18. Find an equation of
   (a) the xy-plane (b) the xz-plane (c) the yz-plane
19. Find an equation of the plane that contains the point (x_0, y_0, z_0) and is
   (a) parallel to the xy-plane
   (b) parallel to the yz-plane
   (c) parallel to the xz-plane
20. Find an equation for the plane that passes through the origin and is parallel to the plane $7x + 4y - 2z + 3 = 0$.
21. Find an equation for the plane that passes through the point $(3, -6, 7)$ and is parallel to the plane $5x - 2y + z - 5 = 0$.
22. Find the point of intersection of the line

$$x - 9 = -5t, \quad y + 1 = -t, \quad z - 3 = t \quad (-\infty < t < +\infty)$$

   and the plane $2x - 3y + 4z + 7 = 0$.
23. Find an equation for the plane that contains the line $x = -1 + 3t$, $y = 5 + 2t$, $z = 2 - t$ and is perpendicular to the plane $2x - 4y + 2z = 9$.
24. Find an equation for the plane that passes through $(2, 4, -1)$ and contains the line of intersection of the planes $x - y - 4z = 2$ and $-2x + y + 2z = 3$.
25. Show that the points $(-1, -2, -3)$, $(-2, 0, 1)$, $(-4, -1, -1)$, and $(2, 0, 1)$ lie in the same plane.

26. Find parametric equations for the line through $(-2, 5, 0)$ that is parallel to the planes $2x + y - 4z = 0$ and $-x + 2y + 3z + 1 = 0$.

27. Find an equation for the plane through $(-2, 1, 5)$ that is perpendicular to the planes $4x - 2y + 2z = -1$ and $3x + 3y - 6z = 5$.

28. Find an equation for the plane through $(2, -1, 4)$ that is perpendicular to the line of intersection of the planes $4x + 2y + 2z = -1$ and $3x + 6y + 3z = 7$.

29. Find an equation for the plane that is perpendicular to the plane $8x - 2y + 6z = 1$ and passes through the points $P_1(-1, 2, 5)$ and $P_2(2, 1, 4)$.

30. Show that the lines

$$x = 3 - 2t, \quad y = 4 + t, \quad z = 1 - t \qquad (-\infty < t < +\infty)$$

and

$$x = 5 + 2t, \quad y = 1 - t, \quad z = 7 + t \qquad (-\infty < t < +\infty)$$

are parallel, and find an equation for the plane they determine.

31. Find an equation for the plane that contains the point $(1, -1, 2)$ and the line $x = t$, $y = t + 1$, $z = -3 + 2t$.

32. Find an equation for the plane that contains the line $x = 1 + t$, $y = 3t$, $z = 2t$ and is parallel to the line of intersection of the planes $-x + 2y + z = 0$ and $x + z + 1 = 0$.

33. Find an equation for the plane, each of whose points is equidistant from $(-1, -4, -2)$ and $(0, -2, 2)$.

34. Show that the line

$$x - 5 = -t, \quad y + 3 = 2t, \quad z + 1 = -5t \qquad (-\infty < t < +\infty)$$

is parallel to the plane $-3x + y + z - 9 = 0$.

35. Show that the lines

$$x - 3 = 4t, \quad y - 4 = t, \quad z - 1 = 0 \qquad (-\infty < t < +\infty)$$

and

$$x + 1 = 12t, \quad y - 7 = 6t, \quad z - 5 = 3t \qquad (-\infty < t < +\infty)$$

intersect, and find the point of intersection.

36. Find an equation for the plane containing the lines in Exercise 35.

37. Find parametric equations for the line of intersection of the planes

(a) $-3x + 2y + z = -5$ and $7x + 3y - 2z = -2$

(b) $5x - 7y + 2z = 0$ and $y = 0$

38. Show that the plane whose intercepts with the coordinate axes are $x = a$, $y = b$, and $z = c$ has equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

provided that a, b, and c are nonzero.

39. Find the distance between the point and the plane.

(a) $(3, 1, -2)$; $x + 2y - 2z = 4$

(b) $(-1, 2, 1)$; $2x + 3y - 4z = 1$

(c) $(0, 3, -2)$; $x - y - z = 3$

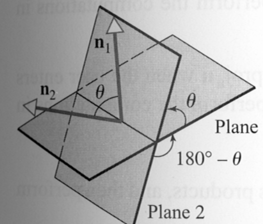

Figure Ex-45

41. Find the distance between the line $x = 3t - 1$, $y = 2 - t$, $z = t$ and each of the following points.

(a) $(0, 0, 0)$ (b) $(2, 0, -5)$ (c) $(2, 1, 1)$

42. Show that if a, b, and c are nonzero, then the line

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \qquad (-\infty < t < +\infty)$$

consists of all points (x, y, z) that satisfy

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These are called ***symmetric equations*** for the line.

43. Find symmetric equations for the lines in parts (a) and (b) of Exercise 9.

Note See Exercise 42 for terminology.

44. In each part, find equations for two planes whose intersection is the given line.

(a) $x = 7 - 4t, \quad y = -5 - 2t, \quad z = 5 + t \qquad (-\infty < t < +\infty)$

(b) $x = 4t, \quad y = 2t, \quad z = 7t \qquad (-\infty < t < +\infty)$

Hint Each equality in the symmetric equations of a line represents a plane containing the line. See Exercise 42 for terminology.

45. Two intersecting planes in 3-space determine two angles of intersection: an acute angle $(0 \le \theta \le 90^\circ)$ and its supplement $180^\circ - \theta$ (see the accompanying figure). If $\mathbf{n}_1$ and $\mathbf{n}_2$ are nonzero normals to the planes, then the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is θ or $180^\circ - \theta$, depending on the directions of the normals (see the accompanying figure). In each part, find the acute angle of intersection of the planes to the nearest degree.

(a) $x = 0$ and $2x - y + z - 4 = 0$

(b) $x + 2y - 2z = 5$ and $6x - 3y + 2z = 8$

Note A calculator is needed.

46. Find the acute angle between the plane $x - y - 3z = 5$ and the line $x = 2 - t$, $y = 2t$, $z = 3t - 1$ to the nearest degree.

Hint See Exercise 45.

Discussion & Discovery

47. What do the lines $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ and $\mathbf{r} = \mathbf{r}_0 - t\mathbf{v}$ have in common? Explain.

48. What is the relationship between the line $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ and the plane $ax + by + cz = 0$? Explain your reasoning.

49. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be vectors from the origin to the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, respectively. What does the equation

$$\mathbf{r} = (1 - t)\mathbf{r}_1 + t\mathbf{r}_2 \qquad (0 \le t \le 1)$$

represent geometrically? Explain your reasoning.

50. Write parametric equations for two perpendicular lines through the point (x_0, y_0, z_0).

51. How can you tell whether the line $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ in 3-space is parallel to the plane $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$?

52. Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) If a, b, and c are not all zero, then the line $x = at$, $y = bt$, $z = ct$ is perpendicular to the plane $ax + by + cz = 0$.

(b) Two nonparallel lines in 3-space must intersect in at least one point.