Homological algebra and algebraic topology
Problem set 1

due: Monday Sept 7 in class.

Problem 1 (2p). Verify that the following sequence is a chain complex and compute its homology groups:

$$\cdots \xrightarrow{6} \mathbf{Z}/12\mathbf{Z} \xrightarrow{4} \mathbf{Z}/12\mathbf{Z} \xrightarrow{6} \mathbf{Z}/12\mathbf{Z} \xrightarrow{4} \mathbf{Z}/12\mathbf{Z} \xrightarrow{6} \cdots$$

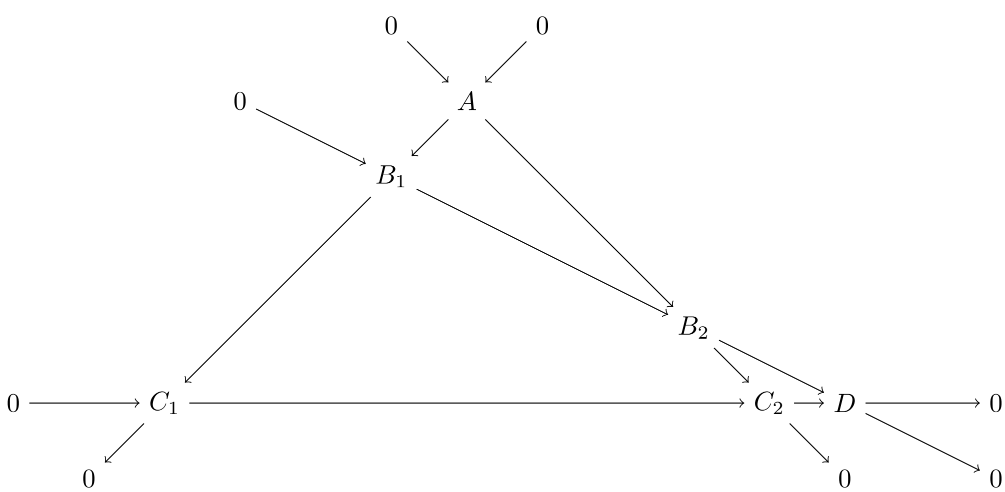
Problem 2 (3p). Let

$$0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$$

be an exact sequence of abelian groups. Show that the following are equivalent:

(1) There exists a homomorphism $q\colon A \to A'$ such that $q \circ i = \mathrm{id}_{A'}$.
(2) There exists a homomorphism $j\colon A'' \to A$ such that $p \circ j = \mathrm{id}_{A''}$.
(3) There is an isomorphism $\phi\colon A \to A' \oplus A''$ such that $\phi \circ i\colon A' \to A' \oplus A''$ is the inclusion into the first coordinate and $p \circ \phi^{-1}\colon A' \oplus A'' \to A''$ is the projection onto the second coordinate.

Problem 3 (3p). Given the following commutative diagram of abelian groups:



Assume that $0 \to A \to B_i \to C_i \to 0$ and $0 \to C_1 \to C_2 \to D \to 0$ are exact. Show, using a diagram chase, that $0 \to B_1 \to B_2 \to D$ is also exact.

Problem 4 (2p). Let $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ be an exact sequence of abelian groups, and let B be another abelian group. Then we obtain a sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(A'', B) & \to & \mathrm{Hom}(A, B) & \to & \mathrm{Hom}(A', B) & \to & 0 \\ & & f & \mapsto & f \circ p & & & & \\ & & & & g & \mapsto & g \circ i. & & \end{array}$$

Is this sequence a chain complex? Is it exact for any choice of A, A', A'', and B?