Homological algebra and algebraic topology
Problem set 10

due: Monday Nov 23 in class.

Problem 1 (4pt)**.** In the following diagram of abelian groups, suppose that the rows are long exact, and the maps γ_n are isomorphisms for all n.

$$\begin{array}{ccccccccc} \cdots C_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{d_n} & \cdots \\ \gamma_{n+1}\downarrow\cong & & \alpha_n\downarrow & & \beta_n\downarrow & & \gamma_n\downarrow\cong & & \\ \cdots C'_{n+1} & \xrightarrow{d'_{n+1}} & A'_n & \xrightarrow{f'_n} & B'_n & \xrightarrow{g'_n} & C'_n & \xrightarrow{d'_n} & \cdots \end{array}$$

Prove that there is a long exact sequence

$$\cdots B'_{n+1} \xrightarrow{d_{n+1}\gamma_{n+1}^{-1}g'_{n+1}} A_n \xrightarrow{(f_n,\alpha_n)} B_n \oplus A'_n \xrightarrow{\beta_n - f'_n} B'_n \xrightarrow{d_n\gamma_n^{-1}g'_n} A_{n-1}\cdots$$

Problem 2 (2pt)**.** Use previous problem to prove that excision for relative homology implies the Mayer - Vietoris axiom.

Problem 3 (2pt)**.** In this problem you will fill in a key step in the proof of excision. Consider a pair of chain complexes of abelian groups $D_* \subseteq C_*$. Suppose that $S\colon C_* \to C_*$ is a chain map that satisfies the following conditions:

(1) $S(D_*) \subseteq D_*$, and both chain maps $S\colon C_* \to C_*$ and $S_*|_{D_*}\colon D_* \to D_*$ induce isomorphisms on all homology groups.
(2) For every $x \in C_n$ there is an m such that $S^m(x) \in D_n$.

Prove that the map $H_n(D_*) \to H_n(C_*)$ induced by the inclusion is an isomorphism for all n.

Problem 4 (2pt)**.** Show that the inclusion of pairs

$$f\colon (D^n, S^{n-1}) \to (D^n, D^n \setminus \{0\}),$$

induces an isomorphism on all relative homology groups. Show that, despite this, f is not a homotopy equivalence of pairs.