Homological algebra and algebraic topology
Problem set 12

due: Monday Dec 7 in class.

Problem 1 (3pt). Let X be a Hausdorff topological space, $x \in X$. The *local homology* of X at x is given by the relative homology groups $\mathrm{H}_*(X, X \setminus \{x\})$.

(1) Prove that for any open neighborhood U of x, $\mathrm{H}_*(X, X \setminus \{x\})$ is isomorphic to $\mathrm{H}_*(U, U \setminus \{*\})$. This explains the term "local".
(2) Let $f \colon X \to X$ be a homeomorphism. Prove that for all $x \in X$, f induces an isomorphism from local homology at x to local homology at $f(x)$.
(3) Let D^n be the closed unit ball in $\mathbb{R}^n$. Calculate the local homology of D^n at interior and boundary points, and use it to prove that every homeomorphism of D^n preserves the boundary.

Problem 2 (3pt). The Klein bottle K is the two-dimensional surface defined as the quotient of the unit square $I \times I$ by the following relations: $(x, 0) \sim (x, 1)$ for all $0 \leq x \leq 1$, and $(0, y) \sim (1, 1-y)$ for all $0 \leq y \leq 1$.

(1) Show that K has a cell structure with one 0-cell, two 1-cells, and one 2-cell.
(2) Write the cellular chain complex of K.
(3) Use it to calculate the homology of K.

Problem 3 (2pt). Let X be the quotient of the sphere S^2 by the identification $x \sim -x$ for every x on the equator of S^2. Calculate the homology of X.

Problem 4 (2pt). Show that if X is a CW complex, then $\mathrm{H}_n(X^n)$ is free.