Homological algebra and algebraic topology

Problem set 13

due: Monday Dec 14 in class.

Problem 1 (3+1pt)**.** Let X be a topological space. Let d_i be the dimension of $H_i(X;\mathbb{Q})$, as a rational vector space. Define the Euler characteristic of X by the formula $\chi(X) = \Sigma_{i=0}^{\infty}(-1)^i d_i$, assuming only finitely many of the d_is are non-zero.

(1) Prove that if X is a finite CW-complex then $\chi(X) = \Sigma_{i=0}^{\infty}(-1)^i c_i$, where c_i is the number of i-dimensional cells of X. (You are allowed to quote a previous homework result if you wish).
(2) Let X be a space, and suppose $X = X_1 \cup X_2$ is a decomposition for which Mayer-Vietoris exact sequence holds. Prove that $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2)$ (assuming that all these numbers are defined).
(3) Let S_1, S_2 be two surfaces. Prove that $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.
(4) (extra credit) Prove that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Problem 2 (2pt)**.** The *projecitve plane* $\mathbb{R}P^2$ is the quotient space of the unit disk by the relation that identifies antipodal points on the boundary.

(1) Show that $\mathbb{R}P^2$ has a CW structure with a single cell in dimensions $0, 1$ and 2. Use it to calculate the homology.
(2) The non-orientable surface of genus g is the g-fold connected sum $\#_g\mathbb{R}P^2$. Describe a cell structure on this space and use it to calculate the homology.

Problem 3 (3pt)**.** Prove that the isomorphism between cellular and singular homology is natural in the following sense. Let $f\colon X \to Y$ be a map of CW complexes. The map f is *cellular* if it has the property that $f(X^n) \subseteq Y^n$ for each n. Prove that if f is cellular then it induces a homomorphism between cellular chain complexes of X and Y, and that the induced map on cellular homology corresponds to the usual induce map on singular homology, under the isomorphism $H_* \cong H_*^{CW}$.

Problem 4 (2pt)**.** Let $I = [0,1]$ and let $A = \{\frac{1}{n} \mid n = 1, 2, \ldots\} \cup \{0\}$. Let X be the CW complex with one 0-cell, and a 1-cell for each positive integer n. All the attaching maps are, necessarily, constant.

(1) Show that there is a continuous bijection $f\colon X \to I/A$.
(2) Show that the map f is not a homeomorphism. Suggestion: find a closed subset of X whose image in I/A is not closed.

Remark: The point of the exercise is to show that the obvious attempt to put a CW structure on I/A fails. It is possible to show that I/A does not have a CW structure.

Problem 5 (Extra credit 2pt)**.** Let I and A be as in the previous exercise. For each positive integer n there is an obvious quotient map $I/A \to [\frac{1}{n+1}, \frac{1}{n}]/\{\frac{1}{n+1}, \frac{1}{n}\}$. In homology, it induces a homomorphism $H_1(I/A) \to \mathbb{Z}$ for each n. Taken together, they give a homomorphism

$$H_1(I/A) \to \prod_{n=1}^{\infty} \mathbb{Z}.$$

Note that the target of this homomorphism is an uncountable group.

(1) Prove that the homomorphism that we just defined is surjective.
(2) Conclude that $H_1(I, A)$ is not isomorphic to $H_1(I/A)$.