# Homological algebra and algebraic topology
# Problem set 14

due: Monday Jan 11 in class.

**Problem 1** (3pt)**.** Identify $S^{2n+1}$ with the space

$$\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = 1\}.$$

I.e., the unit sphere in $\mathbb{C}^{n+1}$. Let $S^1$ be the group of unit complex numbers. It acts on $S^{2n+1}$ by coordinatewise multiplication. The complex projective space $\mathbb{C}P^n$ is the quotient space of this action.

(1) Show that $\mathbb{C}P^n$ has a CW structure with a single cell in dimension $2i$, for $i = 0, 1, \dots, n$.
(2) Use the cell structure to describe the cellular chain complex and the homology of $\mathbb{C}P^n$.

**Problem 2** (2pt)**.** Calculate the homology of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}/2$ and in $\mathbb{Z}/p$, where $p$ is an odd prime.

**Problem 3** (2pt)**.** Suppose that $A \subset S^n$ is a subspace homeomorphic to $S^k \vee S^l$. Prove that the homology of $S^n \setminus A$ is isomorphic to the homology of $S^{n-k-1} \vee S^{n-l-1}$.

**Problem 4** (3pt)**.** Let $X$ be a topological space. Let $X_1 \subset X_2 \subset \cdots$ be a sequence of subsets with the following properties

(1) $X = \bigcup_{n=1}^{\infty} X_n$.
(2) Every compact subspace $A$ of $X$ is contained in $X_n$ for some finite $n$.

Prove the following

(1) Every element of $H_i(X)$ is in the image of $H_i(X_n)$ for some finite $n$.
(2) If an element $u \in H_i(X_1)$ maps to zero in $H_i(X)$, then it maps to zero in $H_i(X_n)$ for some finite $n$.