Homological algebra and algebraic topology
Problem set 2

due: Tuesday Sept 14 in class.

Problem 1 (3pt)**.** Assume that the following diagram of abelian groups has exact rows and columns. Can you determine the missing entries and maps? Give short reasonings.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \quad & \longrightarrow & \quad & \longrightarrow & \mathbf{Z}/2\mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \quad & \longrightarrow & \mathbf{Z} & \longrightarrow & \quad & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z}/3\mathbf{Z} & \longrightarrow & \quad & \longrightarrow & \mathbf{Z}/2\mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Problem 2 (4pt)**.** Let $C_\bullet$ be a chain complex and denote by $C_\bullet/2C_\bullet$ the chain complex which in degree n is the quotient groups $C_n/2C_n$. Consider the sequence

$$0 \to C_\bullet \xrightarrow{2} C_\bullet \to C_\bullet/2C_\bullet \to 0.$$

(1) Show that this sequence is exact if all C_n are free abelian groups.
(2) Give a (preferably small) example of such a $C_\bullet$ that has a nontrivial connecting homomorphism $\delta_n \colon H_{n+1}(C_\bullet/2C_\bullet) \to H_n(C_\bullet)$.

Problem 3 (3pt)**.** Let $\mathcal{C}$ be a category. Denote by $\mathrm{Ar}(\mathcal{C})$ the following category: an object in $\mathrm{Ar}(\mathcal{C})$ is a morphism $X_1 \to X_2$ in $\mathcal{C}$. A morphism in $\mathrm{Ar}(\mathcal{C})$ from $X_1 \to X_2$ to $Y_1 \to Y_2$ is a commutative diagram

$$
\begin{array}{ccc}
X_1 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & Y_2.
\end{array}
$$

(1) Show that $\mathrm{Ar}(\mathcal{C})$ is indeed a category.
(2) Show that $F\colon \mathrm{Ar}(\mathrm{Ab}) \to \mathrm{Ar}(\mathrm{Ab})$ with $F(X_1 \xrightarrow{f} X_2) = (\ker(f) \hookrightarrow X_1)$ defines a functor.