Homological algebra and algebraic topology
Problem set 4

due: Monday Sept 28 in class.

Problem 1 (3pt)**.** Let R be a commmutative ring. An R-algebra is a ring A together with a ring homomorphism

$$\iota\colon R \to Z(A) = \{a \in A \mid ab = ba \text{ for all } b \in A\}.$$

Denote by Alg_R the category of R-algebras and R-algebra homomorphisms. Note that an R-algebra is in particular a left and right R-module by

$$r.a := \iota(r)a = a\iota(r) = a.r$$

Denote by CAlg_R the subcategory of commutative R-algebras.

(1) Show that for A, $B \in \mathrm{CAlg}_R$, the tensor product $A \otimes_R B$ can be given the structure of a commutative R-algebra so that it is the categorical coproduct of A and B in CAlg_R.
(2) Explain why it is in general not a coproduct in Alg_R.

Problem 2 (3pt)**.** Let p be a prime, and denote by $\mathbf{Z}_{(p)} \subset \mathbf{Q}$ the subset of all fractions $\frac{n}{q}$ for which $p \nmid q$.

(1) Show that $\mathbf{Z}_{(p)}$ is a subring of $\mathbf{Q}$.
(2) Show that $\mathbf{Z}_{(p)}$ is flat as a $\mathbf{Z}$-module (abelian group).
(3) Show that if l is another prime and A is a finite abelian l-group then

$$\mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} A \cong \begin{cases} A; & p = l \\ 0; & p \neq l. \end{cases}$$

Problem 3 (2pt)**.** Let R be a ring and M a projective module over R. Show that there exists a free R-module F such that

$$M \oplus F \cong F.$$

Hint: construct F as a suitable infinite direct sum of free modules.

Problem 4 (2pt+2pt)**.** Let R be a ring and M be an R-module. We say that M is **finitely generated** if there is a surjective homomorphism $p\colon R^n \to M$ for some $n \geq 0$. If n and p can be chosen such that $\ker(p)$ is also finitely generated, we call M **finitely presented**.

Denote by M^* the right R-module $\mathrm{Hom}_R(M, R)$, where the right action is given by $(f.r)(m) = f(m)r$.

(1) For R-modules M, N, consider the natural map

$$\begin{aligned} a\colon M^* \otimes_R N &\to \mathrm{Hom}_R(M, N) \\ f \otimes n &\mapsto (m \mapsto f(m)n). \end{aligned}$$

Show that a is an isomorphism if M is finitely presented and projective.
(2) **(bonus +2pt)** Show that finitely presented flat R-modules are projective.