Homological algebra and algebraic topology
Problem set 5

due: Monday October 5 in class.

Problem 1 (3pt)**.** Let R be the polynomial ring $\mathbf{Q}[x, y]$. Construct free resolutions of the following R-modules:

(1) $M = \mathbf{Q}$ with R-action $\mathbf{Q}[x, y] \times \mathbf{Q} \to \mathbf{Q}$ given by $x.a = 0 = y.a$ for all $a \in \mathbf{Q}$.
(2) The ideal $(x, y) \subset R$.

Problem 2 (2pt)**.** Let $R = \mathbf{Z}[x]$ and define a covariant functor $F\colon \mathrm{Mod}_R \to \mathrm{Ab}$ by

$$F(M) = \{x.m \mid m \in M\} \subseteq M.$$

On maps $f\colon M \to N$, we define $F(f)\colon F(M) \to F(N)$ by $F(f) = f|_{F(M)}$. Verify that this is actually a functor. Is F left/right exact?

Problem 3 (3pt)**.** Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be a short exact sequence of R-modules and let $P'_\bullet \to M'$, $P''_\bullet \to M''$ be projective resolutions. Show that there is a projective resolution $P_\bullet \to M$ with $P_n = P'_n \oplus P''_n$ for all $n \geq 0$ such that the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P'_1 \arrow[r, "\iota_1"] \arrow[d] & P_1 \arrow[r, "\pi_2"] \arrow[d] & P''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_0 \arrow[r, "\iota_1"] \arrow[d] & P_0 \arrow[r, "\pi_2"] \arrow[d] & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, "i"] & M \arrow[r, "p"] & M'' \arrow[r] & 0
\end{tikzcd}$$

commutes and has exact rows and columns, where $\iota_1\colon P'_n \to P'_n \oplus P''_n$ denotes the inclusion into the first summand and $\pi_2\colon P'_n \oplus P''_n \to P''_n$ the projection onto the second summand.

Hint: the middle vertical maps are *not* of the form $(x, y) \mapsto (\partial' x, \partial'' y)$!

Problem 4 (2pt)**.** Let R be a ring and $F\colon \mathrm{Mod}_R \to \mathrm{Ab}$ a right exact functor. Using the result of Problem 3 and the homology long exact sequence, show that there is a long exact sequence

$$\cdots \to (L_2F)(M) \xrightarrow{L_2F(p)} (L_2F)(M'') \to (L_1F)(M')$$
$$\xrightarrow{L_1F(i)} (L_1F)(M) \xrightarrow{L_1F(p)} L_1F(M'') \to F(M') \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(M'') \to 0.$$