Homological algebra and algebraic topology
Problem set 8

due: Monday Nov 9, 1pm.

Problem 1 (2pt). Consider the subspaces

$$\Delta_t^{n+1} = \{(t_0, \dots, t_{n+1}) \mid 0 \le t_i \le 1, \quad t_0 + \cdots + t_{n+1} = 1\} \subset \mathbf{R}^{n+2},$$
$$\Delta_x^{n+1} = \{(x_0, \dots, x_n) \mid 0 \le x_0 \le \ldots \le x_n \le 1\} \subset \mathbf{R}^{n+1}.$$

Verify that the change of coordinates $x_i = t_0 + \cdots + t_i$, for $i = 0, 1, \dots, n$, defines a homeomorphism $\Delta_t^{n+1} \to \Delta_x^{n+1}$.

In the next two problems we will use the x-coordinates for Δ^{n+1}.

Problem 2 (3pt). Consider the maps $\eta_0, \dots, \eta_n \colon \Delta^{n+1} \to \Delta^n \times I$ defined by

$$\eta_i(x_0, \dots, x_n) = \big((x_0, \dots, \widehat{x}_i, \dots, x_n), x_i\big).$$

Prove that the simplices $\Delta_i^{n+1} = \mathrm{im}(\eta_i)$ provide a triangulation of $\Delta^n \times I$, i.e.,

$$\Delta^n \times I = \bigcup_{i=0}^{n} \Delta_i^{n+1},$$

and the intersection of Δ_i^{n+1} and Δ_j^{n+1} is either empty or a common face of both. Draw a picture for $n = 2$.

Problem 3 (2pt). In this problem we will fill in the missing step in the proof of homotopy invariance for singular homology. Given a homotopy $h \colon X \times I \to Y$ between f and g, consider the maps $h_0, \dots, h_n \colon S_n X \to S_{n+1} Y$ defined by

$$h_i(\sigma)(x_0, \dots, x_n) = h\big(\sigma(x_0, \dots, \widehat{x}_i, \dots, x_n), x_i\big).$$

Verify the identities

$$\begin{aligned}
d_0 h_0 &= f, \quad d_{n+1} h_n = g, \\
d_i h_j &= h_{j-1} d_i \quad (i < j), \\
d_j h_j &= d_j h_{j-1}, \\
d_i h_j &= h_j d_{i-1} \quad (i > j+1).
\end{aligned}$$

(Hint: Begin by figuring out formulas for the face maps d_i in terms of the x-coordinates for Δ^{n+1}.)

Problem 4 (3pt). Let $\sigma \colon \Delta^1 \to X$ be a singular 1-simplex of X. Define a new 1-simplex $\overline{\sigma}$ by the formula $\overline{\sigma}(x_0, x_1) = \sigma(x_1, x_0)$ (we have reverted to the standard coordinates for Δ^1). Prove that $\sigma + \overline{\sigma}$ is a boundary (i.e., in the image of the boundary homomorphism $\partial_2 \colon C_2(X) \to C_1(X)$).