Homological algebra and algebraic topology
Problem set 9

due: Monday Nov 16 in class.

Problem 1 (3pt+1pt). Let p be a prime and let k be a positive integer. Suppose that X is a topological space with homology groups

$$H_0(X) \cong \mathbf{Z}, \quad H_k(X) \cong \mathbf{Z}/p\mathbf{Z}, \quad H_i(X) = 0, \quad i \neq k.$$

Calculate the cohomology groups $H^i(X)$, $H^i(X; \mathbf{F}_p)$ and $H^i(X; \mathbf{Q})$ for all i.
(**Bonus + 1pt**: can you construct such a space?)

Problem 2 (2pt). Let C be the chain complex $\cdots \to 0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z}$. Let D be the chain complex $\cdots \to 0 \to \mathbb{Z} \to 0$. Let $f\colon C \to D$ be the chain map that is the identity in degree 1 and zero in other degrees. Prove that f is zero on homology, but non-zero on homology with $\mathbb{F}_2$ coefficients. Explain why this means that the splitting in the Universal Coefficients Theorem can not be natural.

Problem 3 (3pt). Let X be a topological space such that $\widetilde{H}_k(X; \mathbf{F}_p) = 0$ for all primes p and all k. Show that the group $\widetilde{H}_k(X)$ is uniquely divisible for every k. (An abelian group A is uniquely divisible if for every $a \in A$ and every non-zero integer n, there is a unique $b \in A$ such that $nb = a$.)

Problem 4 (2pt). Consider a chain complex of finite dimensional vector spaces over a field $\mathbf{k}$,

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0.$$

Let $c_i = \dim_{\mathbf{k}} C_i$ and $h_i = \dim_{\mathbf{k}} H_i(C_*)$. Prove that

$$\sum_{i=0}^{n} (-1)^i c_i = \sum_{i=0}^{n} (-1)^i h_i.$$

(This number is the *Euler characteristic* of the chain complex.)