1. Chain complexes

Definition. A sequence of abelian groups

$$\dots C_{-2},\ C_{-1},\ C_0,\ C_1, \dots$$

with homomorphisms $\partial_i\colon C_{i+1} \to C_i$ is called a (homological) **chain complex** if $\partial_{i-1} \circ \partial_i = 0$ for all $i \in \mathbf{Z}$.

A **cohomological chain complex** is almost the same thing, but with reversed grading: a sequence of abelian groups

$$\dots C^{-2},\ C^{-1},\ C^0,\ C^1, \dots$$

together with homomorphisms $d^i\colon C^{i-1} \to C^i$ such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbf{Z}$.

We will concentrate on homological chain complexes; all results hold analogously for cohomological chain complexes.

A chain complex (or just a sequence of abelian groups with homomorphisms) is called **bounded below** (**bounded above**) if $C_i = 0$ for $i \ll 0$ (resp. $i \gg 0$). It is called **non-negatively graded** (**non-positively graded**) if $C_i = 0$ for $i < 0$ (resp. $i > 0$).

Definition. We call the subgroup $Z_i(C_\bullet) = \ker(\partial_{i-1}\colon C_i \to C_{i-1}) < C_i$ the subgroup of i**-cycles** and the subgroup $B_i(C_\bullet) = \operatorname{im}(\partial_i\colon C_{i+1} \to C_i) < C_i$ the subgroup of i**-boundaries**.

Lemma 1.1. *For any chain complex $C_\bullet$, $B_i(C_\bullet)$ is a subgroup of $Z_i(C_\bullet)$.*

Definition. The ith **homology group** of a chain complex $C_\bullet$ is defined as the quotient group

$$H_i(C_\bullet) = Z_i(C_\bullet)/B_i(C_\bullet).$$

If $Z_n = B_n$ for all n (and thus $H_n = 0$), we call $C_\bullet$ **exact** or **acyclic**. An exact chain complex is more usually called **exact sequence**. An exact sequence of the form

$$0 \to A' \to A \to A'' \to 0$$

is often called a **short exact sequence**.

Lemma 1.2. *Let A, B, C be abelian groups and $f\colon A \to B$ and $g\colon B \to C$ homomorphisms.*

1. $0 \to A \xrightarrow{f} B$ *is exact iff f is injective.*
2. $A \xrightarrow{f} B \to 0$ *is exact iff f is surjective.*
3. $0 \to A \xrightarrow{f} B \to 0$ *is exact iff f is an isomorphism.*
4. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is exact iff f is injective, g is surjective, and* $\ker g = \operatorname{im} f$.

Definition. Let $A_\bullet$, $B_\bullet$ be sequences of abelian groups and homomorphisms (or chain complexes). A **map of sequences** (or **map of chain complexes**) is a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial_n^A} & A_n & \xrightarrow{\partial_{n-1}^A} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial_n^B} & B_n & \xrightarrow{\partial_{n-1}^B} & B_{n-1} & \longrightarrow & \cdots
\end{array}$$

Lemma 1.3. *A map of chain complexes $f\colon C_\bullet \to D_\bullet$ induces maps*

$$\begin{aligned} Z(f)\colon \quad & Z_n(C_\bullet) \to Z_n(D_\bullet) \quad \text{of } n\text{-cycles,} \\ B(f)\colon \quad & B_n(C_\bullet) \to B_n(D_\bullet) \quad \text{of } n\text{-boundaries, and} \\ H_n(f) = f_*\colon \quad & H_n(C_\bullet) \to H_n(D_\bullet) \quad \text{on homology.} \end{aligned}$$

Lemma 1.4 (Five-lemma)**.** *Let*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

be a commutative diagram of abelian groups with exact rows. Then:

(1) *if f_2, f_4 are surjective and f_5 is injective then f_3 is surjective.*
(2) *if f_2, f_4 are injective and f_1 is surjective then f_3 is injective.*
(3) *in particular, if f_1, f_2, f_4, f_5 are isomorphisms then so is f_3.*

Definition. A short exact sequence of the form $0 \to A' \to A' \oplus A'' \to A''$, where the first map is the inclusion into the first summand and the second map is the projection onto the second, is called **split exact**.

See homework problem 1.2 for characterizations of split exact sequences.

Definition. Let $f\colon A \to B$ be a homomorphism between abelian groups. Define its **cokernel** $\operatorname{coker}(f)$ to be the quotient group $B/\operatorname{im}(f)$ and its **coimage** $\operatorname{coim}(f)$ to be $A/\ker(f)$.

Lemma 1.5. *For any homomorphism $f\colon A \to B$ of abelian groups, we have:*

(1) $f\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ *is an isomorphism;*
(2) $0 \to \ker(f) \to A \xrightarrow{f} B \to \operatorname{coker}(f) \to 0$ *is exact.*

Lemma 1.6 (Snake lemma)**.** *Given a diagram of abelian groups*

$$\begin{tikzcd} & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \end{tikzcd} \tag{1.7}$$

with exact rows. Let K_i denote the kernel of $A_i \to B_i$ and C_i its cokernel. Then there is a "snake homomorphism" $K_3 \to C_1$ such that the sequence

$$K_1 \to K_2 \to K_3 \to C_1 \to C_2 \to C_3$$

is exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & K_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & K_3 \arrow[d] \arrow[dddll, out=0, in=180, looseness=2] & \\
 & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[d] & \\
 & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3. \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If $A_1 \to A_2$ is injective then so is $K_1 \to K_2$, and if $B_2 \to B_3$ is injective then so is $C_2 \to C_3$.

Furthermore, the snake map is natural*, meaning that if we have a map $(A_i, B_i) \to (A'_i, B'_i)$ of diagrams of the type (1.7) then the following square commutes:*

$$\begin{tikzcd}
K_3 \arrow[r] \arrow[d] & C_1 \arrow[d] \\
K'_3 \arrow[r] & C'_1.
\end{tikzcd}$$

Theorem 1.8. *Let $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} C_\bullet \to 0$ be a short exact sequence of chain complexes (meaning $0 \to A_n \to B_n \to C_n \to 0$ is exact for each $n \in \mathbf{Z}$). Then there is a* ***connecting homomorphism*** *$\delta_n \colon H_{n+1}(C_\bullet) \to H_n(A_\bullet)$ such that the following long sequence is exact:*

$$\cdots \xrightarrow{p_*} H_{n+1}(C_\bullet) \xrightarrow{\delta_n} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{p_*} H_n(C_\bullet) \xrightarrow{\delta_{n-1}} H_{n-1}(A_\bullet) \xrightarrow{i_*} \cdots .$$

The homomorphism δ is natural: given a map of short exact sequences of chain complexes $(A_\bullet, B_\bullet, C_\bullet) \to (A'_\bullet, B'_\bullet, C'_\bullet)$, the following square commutes:

$$\begin{tikzcd}
H_{n+1}(C_\bullet) \arrow[r, "\delta"] \arrow[d] & H_n(A_\bullet) \arrow[d] \\
H_{n+1}(C'_\bullet) \arrow[r, "\delta"] & H_n(A'_\bullet).
\end{tikzcd}$$

2. Categories and functors

Definition. A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{ob}(\mathcal{C})$ of **objects**;
- for each pair of objects X, $Y \in \mathrm{ob}(\mathcal{C})$, a set of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(X, Y)$;
- for each object $X \in \mathrm{ob}(\mathcal{C})$, an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ called **identity morphism**;

- for each three objects X, Y, $Z \in \mathrm{ob}(\mathcal{C})$, a map
$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}(X,Z), \quad (g,f) \mapsto g \circ f$$
called **composition**.

These have to satisfy the following axioms:

(1) The composition $\circ$ is associative;
(2) For $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $\mathrm{id}_Y \circ f = f$ and $f \circ \mathrm{id}_X = f$.

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is called an **isomorphism** (and the objects X, Y **isomorphic**) if there is another morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that $g \circ f = \mathrm{id}_X$ and $g \circ g = \mathrm{id}_Y$. If such a g exists, it is unique and is denoted by f^{-1}.

We will often abuse notation and write $X \in \mathcal{C}$ for $X \in \mathrm{ob}(\mathcal{C})$, $f \in \mathrm{Hom}(X,Y)$ or even just $f\colon X \to Y$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, and id for id_X. We will also use commutative diagrams to denote equalities between compositions of morphisms.

Definition. We use the following standard notations for familiar categories:

Set**:** The category of sets and functions;
Ab**:** The category of abelian groups and homomorphisms;
Top**:** The category of topological spaces and continuous maps.

Definition. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A **(covariant) functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

- a function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, also called F; and
- for every X, $Y \in \mathrm{ob}(\mathcal{C})$, a function $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ denoted by $f \mapsto F(f)$ or $f \mapsto f_*$

satisfying $(\mathrm{id}_X)_* = \mathrm{id}_{F(X)}$ and $(g \circ f)_* = g_* \circ f_*$.

A **contravariant functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

- a function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, also called F; and
- for every X, $Y \in \mathrm{ob}(\mathcal{C})$, a function $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))$ denoted by $f \mapsto F(f)$ or $f \mapsto f^*$

satisfying $(\mathrm{id}_X)^* = \mathrm{id}_{F(X)}$ and $(g \circ f)^* = f^* \circ g^*$. ("It turns arrows around.")

Definition. A **natural transformation** $\eta\colon F \to G$ between two functors F, $G\colon \mathcal{C} \to \mathcal{D}$ consists of a morphism $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X),G(X))$ for each object $X \in \mathcal{C}$ such that for each morphism $f\colon \mathrm{Hom}_{\mathcal{C}}(X,Y)$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y). \end{tikzcd}$$

Natural transformations between contravariant functors are defined analogously.

A natural transformation $\eta\colon F \to G$ is called **natural isomorphism** (and F and G **isomorphic**, $F \simeq G$) if η_X is an isomorphism for all $X \in \mathcal{C}$.

Definition. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is another functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$, where $\mathrm{Id}_{\mathcal{C}}$, $\mathrm{Id}_{\mathcal{D}}$ denote the identity functors on $\mathcal{C}$ and $\mathcal{D}$, respectively.

Definition. Let $\mathcal{C}$ be a category and $(X_i)_{i\in I}$ a family of objects in $\mathcal{C}$, for some index set I. An object X together with morphisms $\iota_i\colon X_i \to X$ is called **coproduct** of the X_i, and is denoted by $\coprod_{i\in I} X_i$, if for each test object $Y \in \mathcal{C}$, the map

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(\iota_i,-)} \prod_{i\in I} \operatorname{Hom}_{\mathcal{C}}(X_i,Y)$$

is a bijection. The coproduct of only two objects is denoted by $X_1 \sqcup X_2$.

Similarly, an object X with morphism $\pi_i\colon X \to X_i$ is called **product** of the X_i, and is denoted by $\prod_{i\in I} X_i$, if for each test object $Y \in \mathcal{C}$, the map

$$\operatorname{Hom}_{\mathcal{C}}(Y,X) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(-,\pi_i)} \prod_{i\in I} \operatorname{Hom}_{\mathcal{C}}(Y,X_i)$$

is a bijection. The product of only two objects is denoted by $X_1 \times X_2$.

Lemma 2.1. *In an arbitrary category $\mathcal{C}$, (co-)products need not exist, but if they do, they are unique up to isomorphism.*

3. Rings and modules

Definition. A **ring** R is an abelian group together with a **unity** $1 \in R$ and an associative bilinear map $R \times R \to R$, $(x,y) \mapsto xy$, such that $1x = x1 = x$ for all $x \in R$. A ring is called **commutative** if $xy = yx$ for all $x, y \in R$.

A map $f\colon R \to S$ between rings is called a **ring homomorphism** or **map of rings** if it is linear, $f(1_R) = 1_S$, and $f(xy) = f(x)f(y)$ for all $x, y \in R$.

Definition. A **left module** M over a ring R is an abelian group M together with a bilinear multiplication map $R \times M \to M$, $(r,m) \mapsto r.m$, such that $1.m = m$ and $(r_1r_2).m = r_1.(r_2.m)$ for all $m \in M$, $r_i \in R$.

A **right module** is an abelian group M with a bilinear multiplication map $M \times R \to M$, $(m,r) \mapsto m.r$, such that $m.1 = m$ and $m.(r_1r_2) = (m.r_1).r_2$ for all $m \in M$, $r_i \in R$.

When we just say "module", we agree to mean a left module.

A map $f\colon M \to N$ between two (left or right) R-modules M, N is an **R-module homomorphism** if it is a abelian group homomorphism and $f(r.m) = r.f(m)$ (resp. $f(m.r) = f(m).r$) for all $r \in R$, $m \in M$.

The category of left R-modules and R-module homomorphisms is denoted by Mod_R.

Definition. The **product** of a family $(M_i)_{i\in I}$ of R-modules, denoted by $\prod_{i\in I} M_i$, is the module whose underlying abelian group is the product groups, and the R-module structure is given by $r.((m_i)_{i\in I}) = (r.m_i)_{i\in I}$. The **direct sum** of the family, denoted by $\bigoplus_{i\in I} M_i$, is the submodule of families $(m_i)_{i\in I}$ where all but finitely many $m_i = 0$.

An R-module M is called **free** if it is isomorphic to an (arbitrarily indexed) direct sum of copies of R.

Lemma 3.1. *The direct product is a product in* Mod_R *in the category-theoretic sense, and the direct sum is a coproduct.*

Definition. Let R be a ring, M a right R-module, and N a left R-module. The **tensor product** $M \otimes_R N$ is the abelian group obtained as follows. Denote by $\operatorname{Fr}(M \times N)$ the free abelian group with generators pairs (m,n) with $m \in M$, $n \in N$.

Then $M \otimes_R N$ is the quotient of $\mathrm{Fr}(M \times N)$ with respect to an equivalence relation $\sim$ given by:

- $(m_1 + m_2, n) \sim (m_1, n) + (m_2, n)$
- $(m, n_1 + n_2) \sim (m, n_1) + (m, n_2)$
- $(m.r, n) \sim (m, r.n)$

We denote the equivalence class of (m, n) in $M \otimes_R N$ by $m \otimes n$.

Proposition 3.2. *In the context of the previous definition, let T be an abelian group. Denote by $\mathrm{Bil}(M, N; T)$ the set of all bilinear homomorphisms $f \colon M \times N \to T$ with $f(m.r, n) = f(m, r.n)$. Then there is a natural isomorphism*

$$\mathrm{Bil}(M, N; T) \cong \mathrm{Hom}_{\mathbf{Z}}(M \otimes_R N, T).$$

Definition (and lemma)**.** An R-module M is called **projective** if it satisfies the following equivalent conditions:

(1) For each diagram in Mod_R

$$\begin{tikzcd} & M \arrow[dl, dotted] \arrow[d] & \\ N_1 \arrow[r] & N_2 \arrow[r] & 0 \end{tikzcd}$$

with exact row, a lift (dotted arrow) exists such that the resulting diagram commutes.

(2) There is an R-module N such that $M \oplus N$ is free.

(3) Every shot exact sequence $0 \to N_1 \to N_2 \to M \to 0$ splits.

(4) The functor $\mathrm{Hom}_R(M, -)$ maps exact sequences to exact sequences (the functor "is exact").

Lemma 3.3. *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of right R-modules, and let M be a left R-module. Then the sequence of abelian groups*

$$N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$$

is exact. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of left R-modules, and let M be another left R-module. Then the sequence of abelian groups

$$0 \to \mathrm{Hom}_R(N'', M) \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N', M)$$

is exact.

Definition. A left R-module M is called **flat** if the functor $- \otimes_R M$ from right R-modules to abelian groups is exact. A right R-module is flat if the functor $M \otimes_R -$ from left R-modules to abelian groups is exact.

Lemma 3.4. *Free modules are projective. Projective modules are flat. Not every flat module is projective, and not every projective module is free.*

4. Resolutions and derived functors

Definition. Let R be a ring. A nonnegatively graded chain complex $P_\bullet$ of R-modules together with a map $\epsilon P_0 \to M$ (the "augmentation") is called a **projective resolution** of M if

- For every $i \geq 0$, P_i is projective;
- The extended chain complex $\cdots \to P_1 \to P_0 \xrightarrow{\epsilon} M$ is exact.

Proposition 4.1. *Every R-module M has a projective resolution.*

Corollary 4.2. *If R is a principal ideal domain then every R-module has a projective resolution of length 2:*

$$0 \to P_1 \to P_0 \to M \to 0$$

Definition. Let $C_\bullet$, $D_\bullet$ be nonnegatively graded chain complexes of R-modules and let f, $g\colon C_\bullet \to D_\bullet$ be two chain maps. A **chain homotopy** from f to g is a sequence of R-linear maps $h_n\colon C_{n-1} \to D_n$ such that

$$g - f = h \circ \partial^C + \partial^D \circ h.$$

If such a chain homotopy exists, we call f and g **chain homotopic** and write $f \simeq g$.

If $f\colon C_\bullet \to D_\bullet$ and $g\colon D_\bullet \to C_\bullet$ are chain maps with chain homotopies $\mathfrak{g} \circ f \simeq \mathrm{id}_{C_\bullet}$ and $f \circ g \simeq \mathrm{id}_{D_\bullet}$, we call f and g **chain homotopy equivalences** and the chain complexes $C_\bullet$ and $D_\bullet$ **chain homotopy equivalent**.

Proposition 4.3. *If $f \simeq g$ then $f_* = g_*\colon H_*(C_\bullet) \to H_*(D_\bullet)$.*

Theorem 4.4. *Let $f\colon M \to N$ be a morphism of R-modules, $P_\bullet \to M$ a chain complex where all P_i are projective, and $N_\bullet \to N \to 0$ be an exact complex. Then*

(1) *The exists a chain map $f_\bullet\colon P_\bullet \to N_\bullet$ making the following ladder commute:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & N_1 \arrow[r] & N_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

(2) *Any two such extensions $f_\bullet$, $g_\bullet$ are chain homotopic.*

Corollary 4.5. *Any two projective resolutions of M are chain homotopy equivalent.*

Definition. Let R, S be two rings and $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ a (covariant or contravariant) functor. We call F **additive** if the induced map on Hom-sets

$$\mathrm{Hom}_R(M, N) \xrightarrow{F} \mathrm{Hom}_S(F(M), F(N)) \quad (\text{resp. } \mathrm{Hom}_S(F(N), F(M)))$$

is a homomorphism of abelian groups.

Let F be an additive covariant functor as above. Then we call F

- **left exact** if $0 \to F(M') \to F(M) \to F(M'')$ is exact;
- **right exact** if $F(M') \to F(M) \to F(M'') \to 0$ is exact;
- **exact** if it is right and left exact, i. e. if $0 \to F(M') \to F(M) \to F(M'') \to 0$ is exact

for all choices of exact sequences $0 \to M' \to M \to M'' \to 0$ of R-modules.

Similarly, if F is contravariant, we call it

- **left exact** if $0 \to F(M'') \to F(M) \to F(M')$ is exact;
- **right exact** if $F(M'') \to F(M) \to F(M') \to 0$ is exact;
- **exact** if it is right and left exact, i. e. if $0 \to F(M'') \to F(M) \to F(M') \to 0$ is exact

for all choices of exact sequences $0 \to M' \to M \to M'' \to 0$ of R-modules.

Definition (and lemma)**.** Let $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ be a covariant right exact functor, M an R-module, and $P_\bullet \to M$ a projective resolution of M. Define the n**th left derived functor** $L_nF\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ by

$$(L_nF)(N) = H_n(F(P_\bullet)).$$

Similarly, if F is a contravariant left exact functor, define the n **right derived functor** $R^nF\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ by

$$(R^nF)(N) = H^n(F(P_\bullet)).$$

This is independent of the choice of resolution and extends to a functor by defining it on morphisms as follows: if $f\colon M \to M'$ is a morphism of R-modules, extend it to a morphism $f_\bullet\colon P_\bullet \to P'_\bullet$ by Thm. 4.4 and set

$$L_n(F)(f) = H_n(F(f_\bullet));$$

similarly for right derived functors.

Lemma 4.6. *If F is covariant right exact then $L_0F = F$. If F is contravariant left exact then $R^0F = F$.*

Lemma 4.7. *If R is a principal ideal ring and $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ a right exact covariant or left exact contravariant functor. Then $L_nF = 0$ (resp. $R^nF = 0$) if $n \geq 2$.*

Lemma 4.8. *Let F be a covariant left exact functor. Then $L_nF = 0$ for all $n \geq 1$ if and only if F is exact.*

Definition. Let R be a ring, M a right R-module, and N a left R-module. Define $\mathrm{Tor}_n^R(M,N)$ to be the nth left derived functor of the functor $-\otimes_R N\colon {}_R\mathrm{Mod} \to \mathrm{Ab}$, applied to M:

$$\mathrm{Tor}_n^R(M,N) = \left[L_n(-\otimes_R N)\right](M).$$

Let M and N be left modules. Define $\mathrm{Ext}_R^n(M,N)$ to be the nth right derived functor of the functor $\mathrm{Hom}_R(-,N)$, applied to M:

$$\mathrm{Ext}_R^n(M,N) = \left[R^n \mathrm{Hom}(-,N)\right](M)$$

Proposition 4.9. *(symmetry of Tor) The functor Tor_n^R coincides with the nth left derived functor of the functor $M \otimes_R -\colon \mathrm{Mod}_R \to \mathrm{Ab}$, applied to N:*

$$\mathrm{Tor}_n^R(M,N) = \left[L_n(M \otimes_R -)\right](N).$$

5. Homology of spaces

Definition. Denote by Top the category of topological spaces and continuous maps. We also write Top_* for the category of **pointed spaces**. Its objects are pairs (X, x_0) where X is a topological spaces and $x_0 \in X$. Morphisms from (X, x_0) to (Y, y_0) in Top_* are continuous maps $f\colon X \to Y$ such that $f(x_0) = y_0$.

Definition (recollection)**.** Two maps $f,\ g\colon X \to Y$ are called **homotopic** ($f \simeq g$) if there exists a **homotopy** between them, i.e. a map $H\colon X \times [0,1] \to Y$ with $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We call two spaces X and Y **homotopy equivalent** if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

5.1. **Cones, mapping cones, and suspensions.**

Definition. Let X be a space. Its **(unreduced) cone** is the space

$$CX = X \times [0,1]/\sim,$$

where $(x,1) \sim (x',1)$ for all $x,\ x' \in X$. If x_0 is a fixed base point of X, we also denote its **reduced cone** by $C^{\mathrm{red}}X$; it is defined by

$$CX = X \times [0,1]/sim,$$

where $(x,1) \sim (x',1)$ as before but also $(x_0,t) = (x_0,t')$ for all $t,\ t' \in [0,1]$.

Lemma 5.1. *A map $f\colon X \to Y$ is homotopic to a constant map ("null-homotopic") iff it extends to a map $\tilde{f}\colon CX \to Y$ from the unreduced cone on X to Y.*

A pointed map $f\colon (X,x_0) \to (Y,y_0)$ is homotopic to the constant map with value y_0 via a homotopy that does not move x_0 iff it extends to a map $\tilde{f}\colon C^{\mathrm{red}}X \to Y$ from the reduced cone on X to Y.

Definition. Given a map $f\colon A \to X$, define its **(unreduced) mapping cone** by

$$C_f = (A \times [0,1] \sqcup X)/\sim,$$

where $(a,1) \sim (a',1)$ for all $a,\ a' \in A$ and $(a,0) \sim f(a)$ for $a \in A$. Similarly, if f is a pointed map with $f(a_0) = x_0$, the **reduced mapping cone** C_f^{red} is obtained by adding

$$(a_0,t) \sim (a_0,t') \sim x_0$$

to the equivalence relation, for all $t,\ t' \in [0,1]$.

Lemma 5.2. *Let $f\colon A \to X$, $g\colon X \to Y$ be maps. Then g extends to $\tilde{g}\colon C_f \to Y$ iff the composite $g \circ f$ is homotopic to a constant map.*

If all maps are pointed then g extends to $\tilde{g}\colon C_f^{\mathrm{red}} \to Y$ iff the composite $g \circ f$ is homotopic to the constant map with value y_0 via a homotopy that does not move x_0.

Definition. The **unreduced suspension** SX of a space X is the unreduced mapping cone of the unique map $X \to *$; the **reduced suspension** ΣX of a pointed space X is the reduced mapping cone of the unique pointed map $X \to *$.

Remark 5.3. For "good" spaces X and base points $x_0 \in X$, the quotient maps $CX \to C^{\mathrm{red}}X$, $SX \to \Sigma X$, and, for based maps $A \to X$, $C_f \to C_f^{\mathrm{red}}$, are homotopy equivalences. "Good" here means "well-pointed", which is implied for instance if x_0 has a contractible neighborhood in X.

5.2. **The Eilenberg-Steenrod axioms.** Let R be a ring, A an R-module, and

$$H_n\colon \mathrm{Top} \to \mathrm{Mod}_R$$

be a sequence of functors. We write $\tilde{H}_n(X) = \ker(H_n(X) \to H_n(*))$, where the map is induced by the unique map $X \to *$.

Then $(H_n)_{n\in\mathbf{Z}}$ is called a **homology theory with coefficients in** A if the following axioms hold:

- **homotopy:** if $f \simeq g$ then $H_n(f) = H_n(g)$ for all $n \in \mathbf{Z}$.
- **additivity:** if $X = \coprod_{i\in I} X_i$ then $\bigoplus_{i\in I} H_n(X_i) \cong H_n(X)$; the isomorphism is given by the canonical inclusions $X_i \hookrightarrow X$.
- **dimension:** $H_n(*) = \begin{cases} 0; & n \neq 0 \\ A; & n = 0. \end{cases}$ In particular, $H_n(X) \cong \tilde{H}_n(X)$ for $n \neq 0$.

exactness: Let $f\colon A \to X$ be a map and $g\colon X \to C_f$ be the standard inclusion. Then there is a natural long exact sequence

$$\cdots \to H_n(A) \xrightarrow{f_*} H_n(X) \xrightarrow{g_*} \tilde{H}_n(C_f) \to H_{n-1}(A) \to \cdots$$

Mayer-Vietoris: Let $X = U \cup V$, where U and V are open subsets of X, and $Z = U \cap V$. Then there is a long exact sequence

$$\cdots \to H_n(Z) \xrightarrow{i_* - j_*} H_n(U) \oplus H_n(V) \xrightarrow{p_* + q_*} H_n(X) \to H_{n-1}(Z) \to \cdots ,$$

where the map $i\colon Z \hookrightarrow U$, $j\colon Z \hookrightarrow V$, $p\colon U \hookrightarrow X$, $q\colon V \hookrightarrow X$ are all the standard inclusions.

Theorem 5.4. *For every ring R and every R-module A, there exists (up to equivalence of functors) precisely one homology theory with coefficients in A.*

5.3. **Beginning calculations.** For simplicity, let $R = \mathbf{Z}$, $A = \mathbf{Z}$.

Lemma 5.5. *If X is discrete then* $H_n(X) \cong \begin{cases} 0; & n \neq 0 \\ \bigoplus_{x \in X} \mathbf{Z}; & n = 0. \end{cases}$

Lemma 5.6. *Denote by $\mathbf{S}^k$ the standard k-dimensional sphere. Then*

$$\tilde{H}_n(\mathbf{S}^k) \cong \begin{cases} 0; & n \neq k \\ \mathbf{Z}; & n = k. \end{cases}$$

Lemma 5.7. *For any pointed space X, $H_{n+1}(\Sigma X) \cong \tilde{H}_n(X)$.*

Lemma 5.8. *Let $\mathbf{D}^{n+1}$ be the $(n+1)$-dimensional disk, which has $\mathbf{S}^n$ as boundary. There is no continuous function $\mathbf{D}^{n+1} \to \mathbf{S}^n$ which is the identity, or even homotopic to the identity, on $\mathbf{S}^n$.*

Corollary 5.9 (Brouwer's fixed point theorem)**.** *Every continuous self-map of $\mathbf{D}^n$ has a fixed point.*

5.4. **Mapping degrees.** A map $f\colon \mathbf{S}^n \to \mathbf{S}^n$ gives a homomorphism of homology groups $H_n(\mathbf{S}^n) \cong \mathbf{Z}$, so it's multiplication by a number d, called the mapping degree of f, $\deg(f)$.

Lemma 5.10. *If $f\colon \mathbf{S}^n \to \mathbf{S}^n$ is homotopic to a constant map then* $\deg(f) = 0$. □

Lemma 5.11. $\deg(\mathrm{id}) = 1$

Lemma 5.12. $\deg(f \circ g) = \deg(f)\deg(g)$.

Lemma 5.13. *If $f \in O(n+1)$ then* $\deg(f) = \det(f)$.

Corollary 5.14. *The map $x \mapsto -x$ on $\mathbf{S}^n$ has degree $(-1)^{n+1}$. (This map is called the antipodal map.)*

Corollary 5.15. *If $f\colon \mathbf{S}^n \to \mathbf{S}^n$ has no fixed points then* $\deg f = (-1)^{n+1}$.

Theorem 5.16 (Hairy ball theorem)**.** *Let n be even and $f\colon \mathbf{S}^n \to \mathbf{R}^{n+1}$ be a continuous map such that $f(x) \perp x$ for all x. Then $f(x) = 0$ for some $x \in \mathbf{S}^n$.*

6. Singular homology

Definition. The **standard n-simplex** is the topological space

$$\Delta^n = \{(t_0, \dots, t_n) \mid 0 \le t_i \le 1, \quad t_0 + \dots + t_n = 1\} \subset \mathbf{R}^{n+1},$$

topologized as a subspace of $\mathbf{R}^{n+1}$.

A **singular n-simplex** in a topological space X is a continuous map

$$\sigma \colon \Delta^n \to X.$$

The set of singular n-simplices in X is denoted $S_n X$.

The group of **n-chains** is defined to be

$$C_n(X) = \mathbf{Z} S_n X,$$

the free abelian group on the set of singular n-simplices of X. Thus, its elements are formal linear combinations

$$\sum_{\sigma \in S_n X} a_\sigma \sigma,$$

where $a_\sigma \in \mathbf{Z}$, and $a_\sigma = 0$ for all but finitely many σ.

The **boundary homomorphism**

$$\partial_n \colon C_n(X) \to C_{n-1}(X),$$

is defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma),$$

where the **face maps** $d_i \colon S_n X \to S_{n-1} X$ $(i = 0, 1, \dots, n)$, are defined by

$$d_i(\sigma)(t_0, \dots, t_{n-1}) = \sigma(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

The **singular chain complex** $C_*(X)$ is the chain complex

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0.$$

The identity $\partial_n \circ \partial_{n+1} = 0$ follows from the identities $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$.

The **singular homology** of X is defined to be the homology groups of the singular chain complex;

$$H_n(X) = H_n(C_*(X)).$$

Functoriality. Given a continuous map $f \colon X \to Y$, there is an induced chain map $f_* \colon C_*(X) \to C_*(Y)$ defined by $f_*(\sigma) = f \circ \sigma$. Hence, there is an induced homomorphism in homology $f_* \colon H_n(X) \to H_n(Y)$, and this makes $H_n(-)$ into a functor from topological spaces to abelian groups.

6.1. Eilenberg-Steenrod axioms for singular homology.

Theorem 6.1. *Singular homology satisfies the Eilenberg-Steenrod axioms.*

(Dimension) The singular chain complex of a one-point space is isomorphic to

$$\cdots \mathbf{Z} \xrightarrow{=} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{=} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \to 0.$$

The homology groups are clearly $H_0(*) = \mathbf{Z}$ and $H_n(*) = 0$ for $n \neq 0$.

(Additivity) Since Δ^n is connected, every continuous map $\sigma\colon \Delta^n \to \coprod_{i\in I} X_i$ factors through some X_i. This observation can be used to establish an isomorphism of chain complexes
$$C_*(\coprod_{i\in I} X_i) \cong \bigoplus_{i\in I} C_*(X_i),$$
which in turn implies additivity of singular homology.

(Homotopy) Given continuous maps $f, g\colon X \to Y$, and a homotopy $h\colon f \simeq g$, it is possible to construct an explicit chain homotopy $H_n\colon C_n(X) \to C_{n+1}(Y)$ between the induced chain maps $f_*, g_*\colon C_*(X) \to C_*(Y)$. One sets
$$H_n(\sigma) = \sum_{i=0}^{n} (-1)^i h_i(\sigma),$$
where $h_0, \ldots, h_n\colon S_nX \to S_{n+1}Y$ are defined by
$$h_i(\sigma)(x_0, \ldots, x_n) = h(\sigma(x_0 \ldots, \widehat{x}_i, \ldots, x_n), x_i).$$
Here we are using a new set of coordinates for the standard $(n+1)$-simplex;
$$\Delta^{n+1} \cong \{(x_0, \ldots, x_n) \mid 0 \le x_0 \le \ldots \le x_n \le 1\} \subset \mathbf{R}^{n+1},$$
see Homework 8.

(Exactness) Later in the course.

(Mayer-Vietoris) Later in the course.

6.2. **Singular homology and cohomology with coefficients.**

Definition. Let M be an abelian group. Define
$$C_n(X; M) = C_n(X) \otimes_{\mathbf{Z}} M.$$
Then we obtain a chain complex $C_*(X; M)$. The **singular homology of X with coefficients in M** is
$$H_n(X; M) = H_n(C_*(X; M)).$$
Let
$$C^n(X; M) = \mathrm{Hom}_{\mathbf{Z}}(C_n(X), M)$$
be the abelian group of homomorphisms from $C_n(X)$ to M. The **coboundary map**
$$\delta^n\colon C^n(X; M) \to C^{n+1}(X; M)$$
is defined by $\delta^n(f) = f \circ \partial_{n+1}$. We obtain the **singular cochain complex of X with coefficients in M**
$$0 \to C^0(X; M) \xrightarrow{\partial^0} C^1(X; M) \xrightarrow{\partial^1} C^2(X; M) \to \cdots$$
and the **singular cohomology of X with coefficients in M** are the cohomology groups
$$H^n(X; M) = H^n(C^*(X; M)) = \ker \partial^n / \operatorname{im} \partial^{n-1}.$$

Theorem 6.2 (Universal coefficient theorem for homology)**.** *There is a natural short exact sequence*
$$0 \to H_n(X) \otimes_{\mathbf{Z}} M \to H_n(X; M) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{n-1}(X), M) \to 0$$
for every n. The sequence splits, but the splitting is not natural.

Theorem 6.3 (Universal coefficient theorem for cohomology)**.** *There is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{n-1}(X), M) \to H^n(X; M) \to \mathrm{Hom}_{\mathbf{Z}}(H_n(X), M) \to 0$$

for every n. The sequence splits, but the splitting is not natural.

Theorem 6.4. (1) *$\widetilde{H}_n(X) = 0$ for all n if and only if*

$$\widetilde{H}_n(X; \mathbf{Q}) = 0 \quad \textit{and} \quad \widetilde{H}_n(X; \mathbf{F}_p) = 0$$

for all n and all prime numbers p.

(2) *Let $f\colon X \to Y$ be a continuous map. Then*

$$f_*\colon H_n(X) \to H_n(Y)$$

is an isomorphism for all n if and only if

$$f_*\colon H_n(X; \mathbf{Q}) \to H_n(Y; \mathbf{Q}) \quad \textit{and} \quad f_*\colon H_n(X; \mathbf{F}_p) \to H_n(Y; \mathbf{F}_p)$$

are isomorphisms for all n and all prime numbers p.

7. Cell complexes

Recall the definition of the n-disk and the $(n-1)$-sphere;

$$D^n = \{x \in \mathbf{R}^n \mid |x| \leq 1\},$$
$$S^{n-1} = \partial D^n = \{x \in \mathbf{R}^n \mid |x| = 1\}.$$

Definition. A **cell complex** (or **CW-complex**) is a topological space constructed inductively as follows.

- Start with a discrete set of points X^0.
- The **n-skeleton** X^n is obtained from X^{n-1} by attaching n-dimensional cells. More precisely, there is a family of n-disks
$$\{D^n_i\}_{i \in I_n}$$
together with **attaching maps** from their boundaries to X^{n-1},
$$\varphi_i\colon \partial D^n_i \to X^{n-1}, \quad i \in I_n,$$
such that X^n is the quotient space
$$X^n = X^{n-1} \coprod_i D^n_i / \sim$$
where we make the identifications
$$a \sim \varphi_i(a), \quad \text{for} \quad a \in \partial D^n_i.$$
- Finally $X = \cup_n X^n$, where $U \subseteq X$ is open if and only if $U \cap X^n$ is open in X^n for every n.

A **finite dimensional cell complex** is a cell complex X such that $X = X^n$ for some n. The cell complex X is of dimension n if $X = X^n$ but $X \neq X^{n-1}$. A **finite cell complex** is a finite dimensional cell complex that has finitely many cells in each dimension. The **Euler characteristic** of a finite n-dimensional cell complex X is defined as the alternating sum

$$\chi(X) = \sum_{i=1}^{n} (-1)^i c_i,$$

where c_i is the number of i-dimensional cells in X.

7.1. **Cellular homology.**

Definition. Let X be a cell complex. The **cellular chain complex** $C_*^{cell}(X)$,

$$\cdots \to C_n^{cell} \xrightarrow{\partial_n} C_{n-1}^{cell}(X) \to \cdots \to C_1^{cell} \xrightarrow{\partial_1} C_0^{cell}(X) \to 0,$$

has

$$C_n^{cell}(X) = \bigoplus_{i \in I_n} \mathbf{Z} e_i^n,$$

the free abelian group on the n-dimensional cells in X. The differential

$$\partial_n \colon C_n^{cell}(X) \to C_{n-1}^{cell}(X)$$

is defined, on basis elements, by

$$\partial(e_i^n) = \sum_{j \in I_{n-1}} [i:j] e_j^{n-1},$$

where $[i:j] \in \mathbf{Z}$ is the degree of the following self-map of S^{n-1}:

$$S^{n-1} = \partial D_i^n \xrightarrow{\varphi_i} X^{n-1} \to X^{n-1}/\big(X^{n-1} \setminus e_j^{n-1}\big) \cong S^{n-1}.$$

Here $X^{n-1}/\big(X^{n-1} \setminus e_j^{n-1}\big)$ denotes the result of collapsing everything except the cell indexed by $j \in I_{n-1}$ to a point, and φ_i denotes the attaching map of the n-cell indexed by $i \in I_n$. The **cellular homology** of X is the homology of the cellular chain complex:

$$H_n^{cell}(X) = H_n(C_*^{cell}(X)).$$

We can also define cellular homology, or cohomology, with coefficients in an abelian group M as follows:

$$H_n^{cell}(X; M) = H_n(C_*^{cell}(X) \otimes_{\mathbf{Z}} M),$$

and

$$H^n_{cell}(X; M) = H^n\big(\operatorname{Hom}_{\mathbf{Z}}(C_*^{cell}(X), M)\big),$$

respectively.

Theorem 7.1. *For every cell complex X, the cellular homology groups are isomorphic to the singular homology groups,*

$$H_n^{cell}(X) \cong H_n^{sing}(X),$$

for all n.

Corollary 7.2.
- *Cellular homology is independent of the choice of cell decomposition. In fact, it is a homotopy invariant.*
- *For X a finite cell complex, the homology groups (cellular or singular) $H_n(X)$ are finitely generated abelian groups. Moreover, $H_n(X) = 0$ for $n > \dim(X)$.*
- *The Euler characteristic $\chi(X)$ is independent of the cell decomposition. In fact, it is a homotopy invariant, and it may be calculated as*

$$\chi(X) = \sum_{i=0}^{\dim(X)} (-1)^i h_i,$$

where $h_i = \dim_{\Bbbk} H_i(X; \Bbbk)$. Here $\Bbbk$ is any field.

8. Proof of the Eilenberg-Steenrod axioms for singular homology

It remains to prove the Mayer-Vietoris axiom and the Exactness axiom for singular homology.

Definition. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a family of subspaces of a topological space X. The chain complex
$$C_*^{\mathcal{U}}(X) \subseteq C_*(X)$$
is defined as the subcomplex spanned by all singular n-simplices $\sigma\colon \Delta^n \to X$ such that $\mathrm{im}(\sigma) \subseteq U_i$ for some $i \in I$.

Theorem 8.1. *If X is covered by the interiors of the U_i, then the inclusion $C_*^{\mathcal{U}}(X) \to C_*(X)$ is a chain homotopy equivalence.*

This theorem easily implies the following.

Theorem 8.2 (Mayer-Vietoris axiom)**.** *Let $X = U \cup V$, where U and V are open subsets of X. Then there is a long exact sequence*
$$\cdots \to H_n(U\cap V) \xrightarrow{i_*-j_*} H_n(U) \oplus H_n(V) \xrightarrow{p_*+q_*} H_n(X) \to H_{n-1}(U\cap V) \to \cdots,$$

8.1. Relative homology and excision.

Definition. Let $A \subseteq X$ be a subspace. The **relative chain complex** is defined as the quotient chain complex
$$C_*(X, A) = C_*(X)/C_*(A).$$
The **relative homology groups** are the homology groups of the relative chain complex,
$$H_n(X, A) = H_n(C_*(X, A)).$$

By definition, there is a short exact sequence of chain complexes
$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0,$$
and this induces a long exact sequence in homology
$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots.$$

A **pair of topological spaces** is a pair (X, A) where A is a subspace of X. A map of pairs $f\colon (X, A) \to (Y, B)$ is a continuous map $f\colon X \to Y$ such that $f(A) \subseteq B$.

A homotopy between two maps of pairs $f, g\colon (X, A) \to (Y, B)$ is a map of pairs $h\colon (X \times I, A \times I) \to (Y, B)$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$. A map of pairs $f\colon (X, A) \to (Y, B)$ is a homotopy equivalence of pairs if there is a map $g\colon (Y, B) \to (X, A)$ such that the maps of pairs fg and gf are homotopic to the respective identity maps. Relative homology may be viewed as a functor from the category of pairs to the category of abelian groups, and this functor is homotopy invariant in the sense that homotopic maps of pairs induce identical maps in relative homology.

Proposition 8.3. *Given a map of pairs $f\colon (X, A) \to (Y, B)$, there is an induced homomorphism*
$$f_*\colon H_n(X, A) \to H_n(Y, B), \tag{8.4}$$
for every n. If $f_*\colon H_n(X) \to H_n(Y)$ and $(f\,|_A)_*\colon H_n(A) \to H_n(B)$ are isomorphisms for all n, then (8.4) *is an isomorphism for all n.*

Theorem 8.5 (Excision)**.** *Let $Z \subset A \subset X$ be subspaces such that the closure of Z is contained in the interior of A. Then the inclusion of pairs $(X \backslash Z, A \backslash Z) \to (X, A)$ induces an isomorphism in relative homology*

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A)$$

for all n.

Theorem 8.6 (Exactness axiom)**.** *Let $f\colon A \to X$ be a map and $g\colon X \to C_f$ be the standard inclusion into the mapping cone. Then there is a natural long exact sequence*

$$\cdots \to H_n(A) \xrightarrow{f_*} H_n(X) \xrightarrow{g_*} \tilde{H}_n(C_f) \to H_{n-1}(A) \to \cdots$$

The idea of the proof is to look at the long exact sequence in relative homology associated to the pair (M_f, A), where M_f is the **mapping cylinder** of f;

$$M_f = X \coprod A \times I/(a, 0) \sim f(a).$$

By excision and homotopy invariance, we may make the identifications $H_n(M_f) \cong H_n(X)$ and $H_n(M_f, A) \cong \widetilde{H}_n(C_f)$.

9. Real projective spaces

Definition. Real projective n-space $\mathbf{R}\mathrm{P}^n$ is defined to be the set of lines in $\mathbf{R}^{n+1}$ through the origin. For a non-zero vector $x = (x_0, \ldots, x_n) \in \mathbf{R}^{n+1}$, let

$$(x_0 : \ldots : x_n) \in \mathbf{R}\mathrm{P}^n$$

denote the line through 0 and x. The map

$$\pi\colon \mathbf{R}^{n+1} \setminus \{0\} \to \mathbf{R}\mathrm{P}^n$$

$$\pi(x_0, \ldots, x_n) = (x_0 : \ldots : x_n)$$

is surjective, and we give $\mathbf{R}\mathrm{P}^n$ the quotient topology:

$$U \subseteq \mathbf{R}\mathrm{P}^n \text{ open} \iff \pi^{-1}(U) \subseteq \mathbf{R}^{n+1} \setminus \{0\} \text{ open}.$$

Proposition 9.1. *The restriction of π to $S^n \subset \mathbf{R}^{n+1} \setminus \{0\}$ is a quotient map*

$$S^n \xrightarrow{\pi} \mathbf{R}\mathrm{P}^n,$$

and it identifies $\mathbf{R}\mathrm{P}^n$ with the sphere S^n with antipodal points identified. In particular, $\mathbf{R}\mathrm{P}^n$ is compact.

Proposition 9.2. *There is a homeomorphism*

$$\mathbf{R}\mathrm{P}^{n-1} \bigcup_{\pi} D^n \xrightarrow{\cong} \mathbf{R}\mathrm{P}^n\,.$$

In other words, $\mathbf{R}\mathrm{P}^n$ is obtained from $\mathbf{R}\mathrm{P}^{n-1}$ by attaching an n-cell, using the quotient map $\pi\colon S^{n-1} \to \mathbf{R}\mathrm{P}^{n-1}$ as attaching map.

Corollary 9.3. $\mathbf{R}\mathrm{P}^n$ *is an n-dimensional cell complex with one cell in each dimension $0, 1, \ldots, n$. The k-skeleton is $\mathbf{R}\mathrm{P}^k$, where we identify $\mathbf{R}\mathrm{P}^k$ with the subspace of $\mathbf{R}\mathrm{P}^n$ consisting of all points with homogeneous coordinates of the form $(x_0 : \ldots : x_k : 0 : \ldots : 0)$.*

Proposition 9.4. *The map*

$$p\colon \mathbf{R}\mathrm{P}^n \to \mathbf{R}^n \cup \{\infty\}$$

$$(x_0 : \ldots : x_n) \mapsto \Big(\frac{x_0}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\Big),$$

induces a homeomorphism

$$\mathbf{R}\mathrm{P}^n / \mathbf{R}\mathrm{P}^{n-1} \cong \mathbf{R}^n \cup \{\infty\},$$

where the right hand side denotes the one-point compactification of $\mathbf{R}^n$*.*

The one-point compactification of $\mathbf{R}^n$ is the topological space $\mathbf{R}^n \cup \{\infty\}$ obtained by adding a 'point at infinity' ∞, and where the open subsets are the open subsets of $\mathbf{R}^n$ together with all sets of the form $U \cup \{\infty\}$, where $U \subseteq \mathbf{R}^n$ is a set with compact complement.

Choosing a point $p \in S^n$, we can define a map, called **stereographic projection**,

$$s\colon S^n \to \mathbf{R}^n \cup \{\infty\}$$

by declaring $s(p) = \infty$ and, for $x \neq p$, letting $s(x)$ be the point of intersection between the line through x and p and the hyperplane through the origin in $\mathbf{R}^{n+1}$ orthogonal to p (we identify this hyperplane with $\mathbf{R}^n$).

Proposition 9.5. *Stereographic projection defines a homeomorphism* $S^n \cong \mathbf{R}^n \cup \{\infty\}$*.*

By combining the homeomorphisms of Proposition 9.4 and 9.5, we obtain a homeomorphism

$$\mathbf{R}\mathrm{P}^n / \mathbf{R}\mathrm{P}^{n-1} \cong S^n.$$

Proposition 9.6. *The self-map of* S^n *given by the composite*

$$S^n \xrightarrow{\pi} \mathbf{R}\mathrm{P}^n \to \mathbf{R}\mathrm{P}^n / \mathbf{R}\mathrm{P}^{n-1} \cong S^n$$

has degree $1 + (-1)^{n+1}$*.*

Corollary 9.7. *The cellular chain complex of* $\mathbf{R}\mathrm{P}^n$ *may be identified with*

$$0 \to \mathbf{Z}e^n \xrightarrow{1+(-1)^n} \mathbf{Z}e^{n-1} \to \cdots \xrightarrow{2} \mathbf{Z}e^3 \xrightarrow{0} \mathbf{Z}e^2 \xrightarrow{2} \mathbf{Z}e^1 \xrightarrow{0} \mathbf{Z}e^0 \to 0.$$

From this we can read off the homology of $\mathbf{R}\mathrm{P}^n$*:*

$$H_k(\mathbf{R}\mathrm{P}^n) \cong \begin{cases} \mathbf{Z}, & k = 0, \text{ or } k = n \text{ odd}, \\ \mathbf{Z}/2\mathbf{Z}, & 0 < k < n, \ k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

If we take coefficients in $\mathbf{F}_2$*, we get*

$$H_k(\mathbf{R}\mathrm{P}^n; \mathbf{F}_2) \cong \begin{cases} \mathbf{F}_2, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

10. Cohomology ring

Let X be a topological space and let $\Bbbk$ be a ring with unit. We may identify the group of singular cochains $C^n(X;\Bbbk)$ with the set of all functions $f\colon S_nX \to \Bbbk$. For $0 \le i_0 \le \ldots \le i_k \le n$, let

$$(i_0 \cdots i_k)\colon \Delta^k \to \Delta^n$$

denote the linear map that sends e_j to e_{i_j}, for $j = 0, 1, \ldots, k$. For a singular n-simplex $\sigma\colon \Delta^n \to X$, we may compose to get a singular k-simplex $\sigma(i_0 \cdots i_k)\colon \Delta^k \to X$. For instance, $d_i(\sigma) = \sigma(01 \cdots \hat{i} \cdots n)$ in this notation.

We have the singular cochain complex

$$0 \to C^0(X;\Bbbk) \xrightarrow{\delta^0} C^1(X;\Bbbk) \xrightarrow{\delta^1} C^2(X;\Bbbk) \xrightarrow{\delta^2} \cdots,$$

where the coboundary map δ^{n-1} is given by

$$\delta^{n-1}(f)(\sigma) = \sum_{i=0}^{n} (-1)^i f(d_i(\sigma)),$$

for $f \in C^{n-1}(X;\Bbbk)$ and $\sigma \in S_nX$. Recall that the cohomology of X with coefficients in $\Bbbk$ is defined by

$$H^n(X;\Bbbk) = \ker \delta^n / \operatorname{im} \delta^{n-1}.$$

If $f \in \ker \delta^n$, then let $[f] \in H^n(X;\Bbbk)$ denote the cohomology class that f represents.

Definition. The **cup product**

$$C^p(X;\Bbbk) \times C^q(X;\Bbbk) \xrightarrow{\cup} C^{p+q}(X;\Bbbk)$$

is defined by

$$(f \cup g)(\sigma) = f(\sigma(0 \cdots p))g(\sigma(p \cdots p+q)).$$

for $f \in C^p(X;\Bbbk)$, $g \in C^q(X;\Bbbk)$ and $\sigma \in S_{p+q}X$. There is a distinguished 0-cochain $1 \in C^0(X)$, defined by

$$1(\sigma) = 1,$$

for all $\sigma \in S_0X$, where the right hand side denotes the unit element 1 in the ring $\Bbbk$.

Proposition 10.1. *For all $f, f' \in C^p(X;\Bbbk)$, $g, g' \in C^q(X;\Bbbk)$ and $h \in C^r(X;\Bbbk)$, we have*

- $(f + f') \cup g = f \cup g + f' \cup g$ *and* $f \cup (g + g') = f \cup g + f \cup g'$.
- $1 \cup f = f \cup 1 = f$.
- $(f \cup g) \cup h = f \cup (g \cup h)$.
- $\delta(f \cup g) = \delta(f) \cup g + (-1)^p f \cup \delta(g)$.

Theorem 10.2. *The cup product in cohomology*

$$H^p(X;\Bbbk) \times H^q(X;\Bbbk) \xrightarrow{\cup} H^{p+q}(X;\Bbbk),$$

$$([f], [g]) \mapsto [f \cup g],$$

is well-defined and makes

$$H^*(X;\Bbbk) = \bigoplus_{n \ge 0} H^n(X;\Bbbk)$$

into a graded associative ring with unit.

Given a continuous map $\varphi\colon X \to Y$, there is an induced homomorphism

$$\varphi^*\colon C^n(Y;\Bbbk) \to C^n(X;\Bbbk),$$

defined by $\varphi^*(f)(\sigma) = f(\varphi \circ \sigma)$, for $\sigma \in S_n X$. It satisfies the following:

- $\varphi^*(\delta(f)) = \delta(\varphi^*(f))$.
- $\varphi^*(f \cup g) = \varphi^*(f) \cup \varphi^*(g)$.
- $\varphi^*(1) = 1$.

This implies that the map

$$\varphi^*\colon H^*(Y) \to H^*(X)$$

$$[f] \mapsto [\varphi^*(f)]$$

is a well-defined ring homomorphism. Moreover, $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ and $id^* = id$, so we may view cohomology as a contravariant functor $H^*(-)$ from topological spaces to graded rings.

The cross product

$$\times\colon H^p(X;\Bbbk) \times H^q(Y;\Bbbk) \to H^{p+q}(X \times Y;\Bbbk)$$

$$([f],[g]) \mapsto [f \times g]$$

is defined by $(f \times g)(\sigma) = f(\sigma_X(0 \cdots p))g(\sigma_Y(p \cdots p+q))$, for $\sigma \in S_{p+q}(X \times Y)$, where $\sigma = (\sigma_X, \sigma_Y)$.

Theorem 10.3 (Künneth formula)**.** *If $\Bbbk$ is a field and if $H^*(X;\Bbbk)$ or $H^*(Y;\Bbbk)$ is finite dimensional, then the cohomology ring $H^*(X \times Y;\Bbbk)$ is generated by all cross products of elements from $H^*(X;\Bbbk)$ and $H^*(Y;\Bbbk)$.*

Theorem 10.4. *The cohomology ring of projective space $H^*(\mathbf{R}\mathrm{P}^n;\mathbf{F}_2)$ is isomorphic to*

$$H^*(\mathbf{R}\mathrm{P}^n;\mathbf{F}_2) = \mathbf{F}_2\alpha^0 \oplus \mathbf{F}_2\alpha^1 \oplus \mathbf{F}_2\alpha^2 \oplus \cdots \oplus \mathbf{F}_2\alpha^n,$$

where $\alpha^k \in H^k(\mathbf{R}\mathrm{P}^n;\mathbf{F}_2)$ is the unique non-zero element. The cup product is given by

$$\alpha^p \cup \alpha^q = \alpha^{p+q}$$

for $p + q \leq n$, and $\alpha^0 = 1$.

11. The Borsuk-Ulam theorem

Theorem 11.1 (Borsuk-Ulam theorem)**.** *For every continuous map $f\colon S^n \to \mathbf{R}^n$, there is a point $x \in S^n$ such that $f(x) = f(-x)$.*

If we take S^2 as a model for the surface of the earth and if we let $f\colon S^2 \to \mathbf{R}^2$ be the function that measures the temperature and humidity at a given point, then the Borsuk-Ulam theorem tells us that there are always two opposite points on the earth with the exact same temperature and humidity!

Another striking application of the Borsuk-Ulam theorem is the so-called "Ham sandwich theorem".

Theorem 11.2. *Let $A_1, \ldots, A_n$ be compact subsets of $\mathbf{R}^n$. Then there is a hyperplane H in $\mathbf{R}^n$ that simultaneously bisects each of the sets $A_1, \ldots, A_n$.*

Here is some explanation: Every hyperplane H in $\mathbf{R}^n$ is determined by any of its normal vectors n;

$$H = \{x \in \mathbf{R}^n \mid n \cdot x = 0\}.$$

The hyperplane divides any subset $A \subset \mathbf{R}^n$ into two components A^+ and A^-, namely the points $a \in A$ satisfying $n \cdot a > 0$ or $n \cdot a < 0$, respectively. That H bisects the set A means that

$$\mu(A^+) = \mu(A^-) = \frac{1}{2}\mu(A),$$

where μ denotes the standard Lebesgue measure on $\mathbf{R}^n$.

For $n = 3$, if we let A_1, A_2, A_3 be the sets of bread, ham and cheese in a sandwich, then the theorem says that no matter how messily made it is, the sandwich can be cut by a straight cut into two halves with the exact same amount of bread, ham and cheese in each half.

There are many equivalent formulations of the Borsuk-Ulam theorem. We mention two here:

Theorem 11.3. *The following statements are equivalent to the Borsuk-Ulam theorem:*

(1) *There is no antipodal map $g\colon S^n \to S^{n-1}$.*
(2) *For every continuous map $f\colon D^n \to \mathbf{R}^n$ that satisfies $f(-x) = -f(x)$ for all $x \in \partial D^n$, there is a point $x \in D^n$ such that $f(x) = 0$.*

The proof of Theorem 11.3 is left as an assignment (Assignment 14). The proof of the Borsuk-Ulam theorem is a very nice and illustrative application of cup products and the calculation of the cohomology of $\mathbf{R}\mathrm{P}^n$.

Proof of the Borsuk-Ulam theorem. The proof is by contradiction. Assume that there is an antipodal map $f\colon S^n \to S^{n-1}$. Then there is an induced continuous map $\overline{f}\colon \mathbf{R}\mathrm{P}^n \to \mathbf{R}\mathrm{P}^{n-1}$, determined by commutativity of the diagram

(11.4)
$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "\pi"] & S^{n-1} \arrow[d, "\pi"] \\ \mathbf{R}\mathrm{P}^n \arrow[r, "\overline{f}"] & \mathbf{R}\mathrm{P}^{n-1}. \end{tikzcd}$$

The map $\overline{f}$ induces a homomorphism of graded rings

$$\overline{f}^*\colon H^*(\mathbf{R}\mathrm{P}^{n-1}; \mathbf{F}_2) \to H^*(\mathbf{R}\mathrm{P}^n, \mathbf{F}_2).$$

Let α^1 denote the non-zero element in $H^1(\mathbf{R}\mathrm{P}^{n-1}; \mathbf{F}_2)$ and let β^1 denote the non-zero element in $H^1(\mathbf{R}\mathrm{P}^n; \mathbf{F}_2)$ (here we assume $n > 1$).

Lemma 11.5. *We have $\overline{f}^*(\alpha^1) = \beta^1$.*

Assume for the moment the validity of the lemma. By Theorem 10.4, we have that the n-fold cup product of β^1 with itself is

$$\beta^1 \cup \cdots \cup \beta^1 = \beta^n,$$

where β^n is the non-zero element in $H^n(\mathbf{R}\mathrm{P}^n; \mathbf{F}_2)$. On the other hand, if we take the n-fold cup product of α^1, we get

$$\alpha^1 \cup \cdots \cup \alpha^1 = 0,$$

simply because there is nothing in that degree; $H^n(\mathbf{R}\mathrm{P}^{n-1};\mathbf{F}_2)=0$. Now for the punchline: *because* $\overline{f}^*$ *is a ring homomorphism* we must have

$$\begin{aligned} 0=\overline{f}^*(0)=\overline{f}^*(\alpha^1\cup\cdots\cup\alpha^1) &= \overline{f}^*(\alpha^1)\cup\cdots\cup\overline{f}^*(\alpha^1) \\ &= \beta^1\cup\cdots\cup\beta^1 \\ &= \beta^n \end{aligned}$$

where we have used Lemma 11.5 in the middle step. This gives us the contradiction $0=\beta^n$. Thus, our assumption that there exists an antipodal map $f\colon S^n\to S^{n-1}$ must be wrong. □

Proof of Lemma 11.5. It follows from the universal coefficient theorem that if $\mathbf{F}$ is a field, then the first cohomology group $H^1(X;\mathbf{F})$ is just the vector space dual of the first homology group $H_1(X,\mathbf{F})$, for any space X. Thus, we might as well show that the induced map in *homology*

$$\overline{f}_*\colon H_1(\mathbf{R}\mathrm{P}^n;\mathbf{F}_2)\to H_1(\mathbf{R}\mathrm{P}^{n-1},\mathbf{F}_2)$$

is non-zero. We know that both groups are one-dimensional, but how can we describe the generator?

Well, first of all identify Δ^1 with the unit interval $I=[0,1]$. For every path (aka singular 1-simplex) $\gamma\colon I\to S^n$ such that $\gamma(0)$ is antipodal to $\gamma(1)$, we have that the composite $\pi\gamma\colon I\to\mathbf{R}\mathrm{P}^n$ is a loop (because the start and end point get identified), and hence it is a cycle when viewed as an element of the singular chain complex $C_*(\mathbf{R}\mathrm{P}^n;\mathbf{F}_2)$. The associated homology class $[\pi\gamma]\in H_1(\mathbf{R}\mathrm{P}^n;\mathbf{F}_2)$ does not depend on what path γ we use, as long as $\gamma(0)$ is antipodal to $\gamma(1)$, and in fact $[\pi\gamma]$ is the non-zero element. With this description, one can show that $\overline{f}_*\colon H_1(\mathbf{R}\mathrm{P}^n;\mathbf{F}_2)\to H_1(\mathbf{R}\mathrm{P}^{n-1};\mathbf{F}_2)$ is non-zero (see Assignment 14). □