10) For an embedded hypersurface, one has $\partial^2_{ab}\vec{f} = h_{ab}\vec{n} + \frac{1}{2}(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ab}) g^{ec}\partial_c \vec{f}$; $\partial^3_{dab}\vec{f} = \partial^3_{bad}\vec{f}$ requires the Gauß-equations (expressing $h_{ab}h_{de} - h_{ad}h_{be}$ as R_{adbe}, i.e. in terms of $(g_{..})$ and its first and second derivatives) and the Mainardi–Codazzi equations,

$$\partial_d h_{ab} - \partial_b h_{ad} + \Gamma_{ab}^{\;c} h_{cd} + \Gamma_{ad}^{\;c} h_{cb} = 0;$$

prove that $\partial_a \vec{n} = -h_{ab} g^{bc}\partial_c \vec{f}$ is then also consistent, i.e. does not give rise to new conditions on (g_{ab}) and (h_{ab}).

11) Compute the Christoffelsymbols $\Gamma_{ab}^{\;c}$ for an open set of the plane, parametrized by polar coordinates, and [12) calculate the Gauß-curvature K via R_{adbe}]

12') Calculate the value of the Willmore-functional $\int H^2 \sqrt{g}\, du\, dv$ for the torus given in 4) and minimize with respect to the parameters a and b.

13) By first proving that (with $e = \det(e_{ab})$, $X^a e_{ab} Y^b = e(X,Y) = \mathrm{III}(X,Y) = g(LX, LY)$)

$$\int |K|\, dA \left(= \int_B |K| \sqrt{g}\, du\, dv \overset{!}{=} \int_B \sqrt{e}\, du\, dv \right) = A_{S^2}(n(B))$$

$f(B)$

the area swept out on S^2 by the parametrized Gauß-map $n : B \subset U \to S^2$ (which is assumed to be injective, and of maximal rank, on B), derive that at points of non-vanishing Gauß-curvature, $K(p)$ is equal to the (zero-area) limit of the ratio of the (signed) areas on S^2, and $f(U)$ [– which is actually, how Gauß originally introduced (t)his curvature].

14) Give some simple triangulations of a 2-dimensional sphere, a torus, and a pretzel; calculate $V+F-E$, i.e. the respective Euler-Poincaré characteristic(s) χ.

[Verify for a round sphere and the torus given in 4) that, indeed,

$$\int_S K \, dA = 2\pi \chi(S) \quad]$$

15) Prove that the covariant derivative ∇, acting on, and in the direction of, tangential vector fields (along a hypersurface S) satisfies:

$$\nabla_{\varphi X + \psi Z} Y = \varphi \nabla_X Y + \psi \nabla_Z Y \quad \text{(linearity)}$$

$$\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z \quad \text{(additivity)}$$

$$\nabla_X (\varphi Y) = \varphi \nabla_X Y + (\nabla_X \varphi) Y \quad \text{(product rule)}$$

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

(compatibility with the inner product)