1) $\dot{\vec{x}}(t) = \sqrt{2}\,t \begin{pmatrix} -\sin t^2 \\ \cos t^2 \\ 1 \end{pmatrix}$ $\quad \dot{\vec{x}}^2 = 4t^2$

$$L(t) = \int_0^t \sqrt{4\tilde{t}^2}\, d\tilde{t} = \int_0^t 2\tilde{t}\, d\tilde{t} = t^2 = s(t)$$

$$\Rightarrow \vec{r}(s) := \vec{x}(t(s)) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos s \\ \sin s \\ s \end{pmatrix}$$

$$\vec{r}\,' = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin s \\ \cos s \\ 1 \end{pmatrix} \quad \vec{r}\,'' = -\frac{1}{\sqrt{2}} \begin{pmatrix} \cos s \\ \sin s \\ 0 \end{pmatrix} \Rightarrow k = |\vec{r}\,''| = \frac{1}{\sqrt{2}}$$

$$\vec{n} = -\begin{pmatrix} \cos s \\ \sin s \\ 0 \end{pmatrix} \Rightarrow \vec{b} := \vec{r}\,' \times \vec{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin s \\ -\cos s \\ 1 \end{pmatrix}$$

$$\vec{b}\,' = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos s \\ \sin s \\ 0 \end{pmatrix} =: \tau \vec{n} \Rightarrow \tau = -\frac{1}{\sqrt{2}}$$

2) $\ddot{u}^c + \Gamma_{ab}^{\ \ c}(\vec{u}(t))\,\dot{u}^a \dot{u}^b = 0 \quad (\Leftrightarrow \nabla_{\dot{c}}\dot{c} = 0)$

$\downarrow\ \dot{u}^a = v^{a\prime}\dot{\tilde{t}}, \quad \ddot{u}^a = v^{a\prime\prime}\dot{\tilde{t}}^2 + v^{a\prime}\ddot{\tilde{t}}$

$$v^{c\prime\prime} + \Gamma_{ab}^{\ \ c}(\vec{v}(\tilde{t}))\,v^{a\prime} v^{b\prime} = -\frac{\ddot{\tilde{t}}}{\dot{\tilde{t}}^2}\, v'^c \quad \left(\text{i.e. } \nabla_{c'} c' \parallel c'\right)$$

Geodesics on S must be (with u and v as in 4)) of the form $\alpha u(t) + \beta v(t) = \gamma$, $\alpha, \beta, \gamma \in \mathbb{R}$, as S (a half-cylinder of radius $\frac{1}{\sqrt{2}}$) is locally isometric to the plane. See also the solution of 4)

Differentiation of the (parametric) Gauss map, $m = N \circ f : U \subset \mathbb{R}^2 \to S^2$ $\left((u,v) \to \vec{m}(u,v)\right)$

gives $dN\vec{f}_u = \vec{m}_u$, $dN\vec{f}_v = \vec{m}_v$, and

$$\vec{m}_u = a\vec{f}_u + b\vec{f}_v, \quad \vec{m}_v = c\vec{f}_u + d\vec{f}_v \quad (*)$$

then implies that the linear map (dN), as a matrix in the basis $\{\vec{f}_u, \vec{f}_v\}$ is equal to $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. On the other hand, multiplying the Weingarten-equations $(*)$ by $\vec{f}_u$ and $\vec{f}_v$ gives

$$-\underbrace{\begin{pmatrix} \vec{m}\cdot\vec{f}_{uu} & \vec{m}\cdot\vec{f}_{uv} \\ \vec{m}\cdot\vec{f}_{vu} & \vec{m}\cdot\vec{f}_{vv} \end{pmatrix}}_{=:\hat{h}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} \vec{f}_u^{\,2} & \vec{f}_u\vec{f}_v \\ \vec{f}_v\vec{f}_u & \vec{f}_v^{\,2} \end{pmatrix}}_{=:\hat{g}},$$

so that one finds, for the transposed of $-dN$,

$$L^T = -\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \hat{h}\,\hat{g}^{-1} \quad \left(= \left(\sum_{\beta=1}^{2} h_{\alpha\beta}\, g^{\beta\gamma}\right)_{\alpha,\gamma=1,2} \right).$$

[$\vec{m}(u,v) = \vec{N}(f(u,v))$: unit vector, normal to the surface S]

4) $\vec{f}_u = \begin{pmatrix} f'\cos v \\ f'\sin v \\ 1 \end{pmatrix}$, $\vec{f}_v = \begin{pmatrix} f(-\sin v) \\ f\cos v \\ 0 \end{pmatrix}$, $\vec{n} = \frac{\pm 1}{\sqrt{1+f'^2}} \begin{pmatrix} \cos v \\ \sin v \\ -f' \end{pmatrix}$

$\vec{f}_{uu} = \begin{pmatrix} f''\cos v \\ f''\sin v \\ 0 \end{pmatrix}$, $\vec{f}_{vv} = \begin{pmatrix} -f\cos v \\ -f\sin v \\ 0 \end{pmatrix}$, $\vec{f}_{uv} = \begin{pmatrix} -f'\sin v \\ f'\cos v \\ 0 \end{pmatrix}$

$\hat{g} = \begin{pmatrix} f'^2+1 & 0 \\ 0 & f^2 \end{pmatrix}$, $\hat{h} = \frac{\pm 1}{\sqrt{1+f'^2}} \begin{pmatrix} f'' & 0 \\ 0 & -f \end{pmatrix}$

$$\hat{h}\hat{g}^{-1} = \frac{\pm 1}{\sqrt{1+f'^2}} \begin{pmatrix} \frac{f''}{1+f'^2} & 0 \\ 0 & -\frac{1}{f} \end{pmatrix} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix};$$

mean curvature $= \frac{1}{2}(\varkappa_1 + \varkappa_2) \overset{!}{=} 0 \Leftrightarrow f f'' = 1 + f'^2$

$\Leftrightarrow f(u) = \frac{1}{c}\cosh(cu+d)$.

$$\Gamma^v_{uv} = \frac{1}{2} g^{va}\left(\partial_u g_{av} + \partial_v g_{au} - \partial_a g_{uv}\right) = \frac{1}{2} g^{vv} \partial_u g_{vv}$$

$= f'/f$ (and all other $\Gamma^v_{\cdot\cdot} = 0$, as $g_{uv} = 0, \ldots$)

(For a cylinder $f = \text{const.}$, and the geodesic equation in 2) therefore reads $\ddot{v} = 0$, implying $v(t) = at + b$; as a geodesic $\vec{c}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos v(t) \\ \sin v(t) \\ u(t) \end{pmatrix}$ must have $\dot{\vec{c}}^2 = \frac{1}{2}(\dot{v}^2 + \dot{u}^2) \overset{!}{=} \text{const}$, also $\dot{u}^2 = \text{const}$, confirming $\alpha u + \beta v = \gamma$), but $\vec{x}(t)$ in 1) is such a geodesic only after reparametrisation

5) $0 \overset{!}{=} \vec{m} \cdot \left(\partial_d \partial^2_{ab} \vec{f} - \partial_b \partial^2_{ad} \vec{f} \right)$

$= \partial_d h_{ab} + \Gamma^c_{ab} h_{cd} - \partial_b h_{ad} - \Gamma^c_{ad} h_{cb}$

6) cp. www.mat.univie.ac.at/~michor/...
www.math.miami.edu/~larsa/MTH551/Notes/notes.pdf
(see also e-mail from March 21)

7A) $\hat{\mathbb{M}}$, being the union of two trivial manifolds (each isomorphic to the open interval $(-1,+1) \subset \mathbb{R}$) is a manifold (not connected, though).
The charts $(\varphi^{-1}(\mathcal{U}), \varphi)$, $(\tilde{\varphi}^{-1}(\tilde{\mathcal{U}}), \tilde{\varphi})$ with

$\varphi : p \in \{[(x \leq 0, +1)]\} \cup \{[x > 0]\} \to x \in (-1,+1) =: \mathcal{U}$

$\tilde{\varphi} : p \in \{[(\tilde{x} \leq 0, -1)]\} \cup \{[\tilde{x} > 0]\} \to \tilde{x} \in (-1,+1) =: \tilde{\mathcal{U}}$,

$\varphi \circ \tilde{\varphi}^{-1}\big|_{(0,1)} = id = \tilde{\varphi} \circ \varphi^{-1}\big|_{(0,1)} : (0,+1) \to (0,+1)$,

open sets defined as preimages of open intervals, would provide a (C^∞) atlas for $\mathbb{M}$, but as $0^+ := [(0,+1)]$ and $0^- := [(0,-1)]$ can not be separated by any open neighborhoods, $\mathbb{M}$ (as a topological space) is not Hausdorff, thus (according to our definition) not a manifold.

7B) Simplest when assuming the Gauß consistency equations,

$$R(X,Y)Z = h(Z,Y)LX - h(Z,X)LY$$

(the r.h.s. trivially satisfies that the cyclic sum $= 0$)

7C) $\tilde{x}^a = \frac{4x^a}{\vec{x}^2} \Rightarrow \frac{\partial \tilde{x}^a}{\partial x^b} = \frac{4}{\vec{x}^2}\left(\delta^a_b - 2\frac{x^a x_b}{\vec{x}^2}\right)$;

hence on the equator ($\vec{x}^2 = 4$), $\tilde{X}^a_{(\tilde{x}(x))} = \frac{\partial \tilde{x}^a}{\partial x^b} X^b_{(x)}$

reads $\tilde{X}^a = \left(\delta^a_b - \frac{1}{2}x^a x_b\right)X^b =: S^a_b(x) X^b$,

i.e. (with $x^1 = 2\cos\varphi$, $x^2 = 2\sin\varphi$)

$$S = \begin{pmatrix} 1-2\cos^2\varphi & -2\cos\varphi\sin\varphi \\ -2\cos\varphi\sin\varphi & 1-2\sin^2\varphi \end{pmatrix} = \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(a reflection, together with a rotation by 2φ; note that $\tilde{x}^a = x^a$ on the equator, "but" $S \neq 1$)

7D) $\nabla_a g^{ab} \nabla_b f = \nabla_a \partial^a f = \partial_a (g^{ab}\partial_b f) + \Gamma^a_{ac}\partial^c f$

$$\frac{1}{\sqrt{g}}\partial_a \sqrt{g}\, g^{ab}\partial_b f = \partial_a (g^{ab}\partial_b f) + \left(\frac{1}{\sqrt{g}}\partial_c \sqrt{g}\right)\partial^c f$$