

$$\partial_a \vec{m} = -\lambda_{ab} g^{bc} \partial_c \vec{f} \quad , \quad \partial_{ab}^2 \vec{f} = \lambda_{ab} \vec{m} + \Gamma_{ab}^c \partial_c \vec{f}$$

$$\Rightarrow (\vec{0} =) \quad \partial_{ab}^3 \vec{f} - \partial_{bad}^3 \vec{f}$$

$$= \partial_a (\lambda_{ab} \vec{m} + \Gamma_{ab}^c \partial_c \vec{f}) - \partial_b (\lambda_{ad} \vec{m} + \Gamma_{ad}^c \partial_c \vec{f})$$

$$= \vec{m} \partial_a \lambda_{ab} + \lambda_{ab} \partial_a \vec{m} + (\partial_a \Gamma_{ab}^c) \partial_c \vec{f} + \Gamma_{ab}^c \partial_a^2 \vec{f}$$

$$= \vec{m} (\partial_a \lambda_{ab} - \partial_b \lambda_{ad} + \Gamma_{ab}^c \lambda_{cd} - \Gamma_{ad}^c \lambda_{cb}) - (b \leftrightarrow d)$$

$$- \lambda_{ab} \lambda_{cde} g^{ec} \partial_c \vec{f} + \lambda_{ad} \lambda_{bce} g^{ec} \partial_c \vec{f}$$

$$+ (\partial_a \Gamma_{ab}^c - \partial_b \Gamma_{ad}^c + \Gamma_{ab}^e \Gamma_{de}^c - \Gamma_{ad}^e \Gamma_{be}^c) \partial_c \vec{f} \stackrel{!}{=} \vec{0}$$

$$\Rightarrow \partial_a \lambda_{ab} - \partial_b \lambda_{ad} + \Gamma_{ab}^c \lambda_{cd} - \Gamma_{ad}^c \lambda_{cb} = 0$$

(Mainardi-Codazzi equations)

$$(\lambda_{ab} \lambda_{cde} - \lambda_{ad} \lambda_{bce}) g^{ce}$$

$$= \partial_a \Gamma_{ab}^c - \partial_b \Gamma_{ad}^c + \Gamma_{ab}^e \Gamma_{de}^c - \Gamma_{ad}^e \Gamma_{be}^c =: R_{adba}^c$$

(Goursat-equations)

$$\Gamma_{ab}^c = \frac{1}{2} g^{ce} (\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ab})$$

$$\partial_a \vec{m} = -\lambda_{ab} g^{bc} \partial_c \vec{f} \quad , \quad \partial_{ab}^2 \vec{f} = \lambda_{ab} \vec{m} + \Gamma_{ab}^c \partial_c \vec{f}$$

$$\Rightarrow (\vec{0} = ) \quad \partial_{[ab}^3 \vec{f} - \partial_{ba}^3 \vec{f}$$

$$= \partial_d (\lambda_{ab} \vec{m} + \Gamma_{ab}^c \partial_c \vec{f}) - \partial_b (\lambda_{ad} \vec{m} + \Gamma_{ad}^c \partial_c \vec{f})$$

$$= \vec{m} \partial_d \lambda_{ab} + \lambda_{ab} \partial_d \vec{m} + (\partial_d \Gamma_{ab}^c) \partial_c \vec{f} + \Gamma_{ab}^c \partial_d^2 \vec{f}$$

$$= \vec{m} (\partial_d \lambda_{ab} - \partial_b \lambda_{ad} + \Gamma_{ab}^c \lambda_{cd} - \Gamma_{ad}^c \lambda_{cb}) - (b \leftrightarrow d)$$

$$- \lambda_{ab} \lambda_{cd} g^{ec} \partial_e \vec{f} + \lambda_{ad} \lambda_{bc} g^{ec} \partial_e \vec{f} \\ + (\partial_d \Gamma_{ab}^c - \partial_b \Gamma_{ad}^c + \Gamma_{ab}^e \Gamma_{de}^c - \Gamma_{ad}^e \Gamma_{be}^c) \partial_e \vec{f} \stackrel{!}{=} \vec{0}$$

$$\Rightarrow \partial_d \lambda_{ab} - \partial_b \lambda_{ad} + \Gamma_{ab}^c \lambda_{cd} - \Gamma_{ad}^c \lambda_{cb} = 0$$

(Mainardi-Codazzi equations)

$$(\lambda_{ab} \lambda_{cd} - \lambda_{ad} \lambda_{cb}) g^{ce}$$

$$= \partial_d \Gamma_{ab}^c - \partial_b \Gamma_{ad}^c + \Gamma_{ab}^e \Gamma_{de}^c - \Gamma_{ad}^e \Gamma_{be}^c =: R_{adb}^c$$

(Gauss-equations)

$$\Gamma_{ab}^c = \frac{1}{2} g^{ce} (\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ab})$$