$$\partial_a \vec{n} = -h_{ab}\, g^{bc} \partial_c \vec{f}\,, \quad \partial^2_{ab} \vec{f} = h_{ab} \vec{n} + \Gamma_{ab}{}^c \partial_c \vec{f}$$

$$\Rightarrow \left(\vec{0} = \right) \partial^3_{dab} \vec{f} - \partial^3_{bad} \vec{f}$$

$$= \partial_d \left(h_{ab} \vec{n} + \Gamma_{ab}^c \partial_c \vec{f} \right) - \partial_b \left(h_{ad} \vec{n} + \Gamma_{ad}^c \partial_c \vec{f} \right)$$

$$= \vec{n}\, \partial_d h_{ab} + h_{ab} \partial_d \vec{n} + \left(\partial_d \Gamma_{ab}^c \right) \partial_c \vec{f} + \Gamma_{ab}^c \partial^2_{dc} \vec{f} \quad - (b \leftrightarrow d)$$

$$= \vec{n} \left(\partial_d h_{ab} - \partial_b h_{ad} + \Gamma_{ab}^c h_{cd} - \Gamma_{ad}^c h_{cb} \right)$$

$$- h_{ab} h_{de}\, g^{ec} \partial_c \vec{f} + h_{ad} h_{be}\, g^{ec} \partial_c \vec{f}$$

$$+ \left(\partial_d \Gamma_{ab}^c - \partial_b \Gamma_{ad}^c + \Gamma_{ab}^e \Gamma_{de}^c - \Gamma_{ad}^e \Gamma_{be}^c \right) \partial_c \vec{f} \stackrel{!}{=} \vec{0}$$

$$\Rightarrow \partial_d h_{ab} - \partial_b h_{ad} + \Gamma_{ab}^c h_{cd} - \Gamma_{ad}^c h_{cb} = 0$$

(Mainardi-Codazzi equations)

$$\left(h_{ab} h_{de} - h_{ad} h_{be} \right) g^{ec}$$

$$= \partial_d \Gamma_{ab}^c - \partial_b \Gamma_{ad}^c + \Gamma_{ab}^e \Gamma_{de}^c - \Gamma_{ad}^e \Gamma_{be}^c =: R_{adb}{}^c$$

(Gauß-equations)

$$\Gamma_{ab}^c = \frac{1}{2} g^{ce} \left(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ab} \right)$$

$$\partial_a \vec{n} = -h_{ab}\, g^{bc}\, \partial_c \vec{f}\,, \quad \partial^2_{ab}\vec{f} = h_{ab}\vec{n} + \Gamma_{ab}{}^{c}\, \partial_c \vec{f}$$

$$\Rightarrow \left(\vec{0} = \right) \partial^3_{dab}\vec{f} - \partial^3_{bad}\vec{f}$$

$$= \partial_d\left(h_{ab}\vec{n} + \Gamma_{ab}^{c}\partial_c\vec{f}\right) - \partial_b\left(h_{ad}\vec{n} + \Gamma_{ad}^{c}\partial_c\vec{f}\right)$$

$$= \vec{n}\,\partial_d h_{ab} + h_{ab}\,\partial_d\vec{n} + \left(\partial_d\Gamma_{ab}^{c}\right)\partial_c\vec{f} + \Gamma_{ab}^{c}\,\partial^2_{dc}\vec{f} - (b \Leftrightarrow d)$$

$$= \vec{n}\left(\partial_d h_{ab} - \partial_b h_{ad} + \Gamma_{ab}^{c} h_{cd} - \Gamma_{ad}^{c} h_{cb}\right)$$

$$- h_{ab} h_{de}\, g^{ec}\,\partial_c\vec{f} + h_{ad} h_{be}\, g^{ec}\,\partial_c\vec{f}$$

$$+ \left(\partial_d\Gamma_{ab}^{c} - \partial_b\Gamma_{ad}^{c} + \Gamma_{ab}^{e}\Gamma_{de}^{c} - \Gamma_{ad}^{e}\Gamma_{be}^{c}\right)\partial_c\vec{f} \overset{!}{=} \vec{0}$$

$$\Rightarrow \partial_d h_{ab} - \partial_b h_{ad} + \Gamma_{ab}^{c} h_{cd} - \Gamma_{ad}^{c} h_{cb} = 0$$

(Mainardi-Codazzi equations)

$$\left(h_{ab} h_{de} - h_{ad} h_{be}\right) g^{ec}$$

$$= \partial_d\Gamma_{ab}^{c} - \partial_b\Gamma_{ad}^{c} + \Gamma_{ab}^{e}\Gamma_{de}^{c} - \Gamma_{ad}^{e}\Gamma_{be}^{c} =: R_{adb}{}^{c}$$

(Gauß-equations)

$$\Gamma_{ab}^{c} = \frac{1}{2} g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ab}\right)$$