1. $n=1: \quad \frac{2}{1\cdot 2} = 2 - \frac{2}{1+1}$ OK

$n \Rightarrow n+1: \quad \sum_{k=1}^{n+1} \frac{2}{k(k+1)} = \sum_{k=1}^{n} \frac{2}{k(k+1)} + \underbrace{\frac{2}{(n+1)(n+2)}}_{(k=n+1)}$

$= \left(2 - \frac{2}{n+1}\right) + \frac{2}{(n+1)(n+2)} = 2 - \frac{2}{n+1}\left(\underbrace{1 - \frac{1}{n+2}}_{= \frac{n+1}{n+2}}\right)$

$= 2 - \frac{2}{n+2}$.

2. $f(x) := \frac{x}{x^2-5x+4} = \frac{x}{(x-4)(x-1)} \overset{!}{=} \frac{B}{x-4} + \frac{A}{x-1}$

$= \frac{B(x-1)+A(x-4)}{(x-4)(x-1)}$

$\Rightarrow A+B=1, \quad 4A+B=0$

$\Rightarrow A = -\frac{1}{3}, \quad B = \frac{4}{3}$

$\Rightarrow \int_2^3 f(x)\,dx = \frac{4}{3}\int_2^3 \frac{dx}{x-4} - \frac{1}{3}\int_2^3 \frac{dx}{x-1} = \frac{1}{3}\Big(4\ln|x-4| - \ln|x-1|\Big)\Big|_2^3$

$= \frac{1}{3}\left(4\ln 1 - 4\ln 2 - \ln 2 + \ln 1\right) = -\frac{5}{3}\ln 2$ (<0)

3. $g(x) := \frac{1-x^7}{1-x} \underset{x\neq 1}{=} 1 + x + x^2 + \dots + x^6$ (geometrisk serie) $\Rightarrow \lim_{x\to 1} g(x) = \underbrace{1+\dots+1}_{7\text{ termer}} = 7$

Eller Polynomdivision: $x^7-1 : x-1 = x^6 + x^5 + \dots + 1$

Eller $x = 1-t$, $\lim_{x\to 1} g(x) = \lim_{t\to 0} \frac{1-(1-t)^7}{t}$

A (Binomi) $(1-t)^7 = 1 - 7t + O(t^2) \Rightarrow \dots$

B (Taylor) $h(t) := (1-t)^7 = h(0) + t\,h'(0) + O(t^2)$
$= \dots$

4. $\int_0^{\pi/2} \cos x (\sin x)^3 dx \underset{\substack{u=\sin x \\ du = \cos x\, dx}}{=} \int_0^1 u^3 du = \frac{1}{4} u^4 \Big|_0^1 = \frac{1}{4}$

5. $\ldots = \sin(\pi/2) = 1$

6. $y'' - 6y' + 5y = 0 \quad (*) \longrightarrow \lambda^2 - 6\lambda + 5 = 0$

$y = e^{\lambda x} \qquad \lambda_\pm = 3 \pm \sqrt{9-5}$

$\Rightarrow y_h(x) = A e^{5x} + B e^{x}$

$y_1'' - 6y_1' + 5y_1 \overset{!}{=} 4x \longrightarrow -6a + 5ax + 5b = 4x$

$y_1(x) = ax + b \qquad \Rightarrow 5b - 6a = 0, \ b = \frac{24}{25}$

$y_1' = a, \ y_1'' = 0 \qquad 5a = 4, \ a = \frac{4}{5}$

$\Rightarrow y_1(x) = \frac{4}{5}x + \frac{6}{5}$ är en partikulär lösning

$y_2'' - 6y_2' + 5y_2 \overset{!}{=} e^{2x} \longrightarrow e^{2x}(4 - 6\cdot 2 + 5)C \overset{!}{=} e^{2x}$

$y_2(x) = Ce^{2x} \qquad \Rightarrow -3C = 1, \ C = -\frac{1}{3}$

$y_2(x) = -\frac{1}{3} e^{2x}$

$\Rightarrow$ den allmänna lösningen till $(*)$ är

$$y(x) = A \cdot e^{5x} + B \cdot e^{x} - \frac{1}{3} e^{2x} + \frac{4}{25}(5x + 6)$$

7. $\cos 3x + i \sin 3x = e^{3ix} = (e^{ix})^3 = (\cos x + i \sin x)^3$

$\underset{\text{(Binomi)}}{=} (\cos x)^3 + 3i \cos^2 x \sin x - 3 \cos x (\sin x)^2 - i (\sin x)^3$

$= \left((\cos x)^3 - 3\cos x (1 - \cos^2 x)\right) + i\left(3(1 - \sin^2 x)\sin x - (\sin x)^3\right)$

8. $$\int_{-2}^{\infty}\frac{dx}{x^2+6x+10}=\lim_{\Lambda\to\infty}\int_{-2}^{\Lambda}\frac{dx}{(x+3)^2+1}=\lim_{\Lambda\to\infty}\int_{1}^{\Lambda+3}\frac{du}{u^2+1}$$

$$=\lim_{\Lambda\to\infty}\arctan u\Big|_1^{\Lambda+3}=\lim_{\Lambda\to\infty}\left(\arctan(\Lambda+3)-\arctan 1\right)$$

$$=\frac{\pi}{2}-\frac{\pi}{4}=\pi/4>0$$

$$\int_0^1 \underset{h'}{\underset{\uparrow}{x^{\frac{1}{3}}}}\,\underset{g}{\underset{\uparrow}{\ln x}}=\lim_{\epsilon\to 0}\left\{\frac{3}{4}x^{4/3}\ln x\Big|_\epsilon^1-\frac{3}{4}\int_\epsilon^1 \underbrace{x^{\frac{4}{3}}(\ln x)'}_{=x^{\frac{1}{3}}=\left(\frac{3}{4}x^{4/3}\right)'}dx\right\}$$

$h(x)=\frac{3}{4}x^{\frac{4}{3}}$

$$=-\frac{3}{4}\lim_{\epsilon\to 0}\left(\epsilon^{\frac{4}{3}}\ln\epsilon+\frac{3}{4}x^{4/3}\Big|_\epsilon^1\right)$$

$$=-\frac{3}{4}\left(0+\frac{3}{4}-0\right)=-9/16<0$$

9. $$f'(x)=-x^6+4x^5-4x^4=-x^4(x^2-4x+4)=\underset{\leq 0}{-x^4(x-2)^2}$$

$\overset{!}{=}0 \Rightarrow x=0$ eller $x=2$
(båda randpunkter!)

$\overset{f'\leq 0}{\Longrightarrow}$ $x=0$ Maximipunkt, $x=2$ Minimipunkt

$f(0)=0$, $f(2)=2^6\underbrace{\left(-\frac{2}{7}+\frac{2}{3}-\frac{2}{5}\right)}_{=\frac{-2}{7\cdot 3\cdot 5}}<0$

10. $$\int_{t_1}^{t_2}\sqrt{\dot{x}^2+\dot{y}^2}\,dt \underset{\tilde{t}=f(t)}{=}\int_{\tilde{t}_1=f(t_1)}^{\tilde{t}_2=f(t_2)}\sqrt{\dot{\tilde{x}}^2+\dot{\tilde{y}}^2}\,d\tilde{t}$$

$$d\tilde{t}=\dot{f}(t)dt\to dt=d\tilde{t}/\dot{f}$$

$$\dot{x}(t)=\frac{d}{dt}(\tilde{x}(f(t)))=\dot{\tilde{x}}\cdot\dot{f} \text{ (kedjeregel)}$$

$$\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{(\dot{\tilde{x}}^2+\dot{\tilde{y}}^2)\dot{f}^2}=|\dot{f}|\sqrt{\dot{\tilde{x}}^2+\dot{\tilde{y}}^2}\underset{\dot{f}>0}{=}\dot{f}\cdot\sqrt{\dot{\tilde{x}}^2+\dot{\tilde{y}}^2}$$