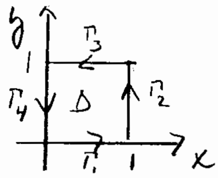
FÖRSLAG TILL LÖSNING. MATEMATIK II, FÖR LÄRARE

2004-05-27. 5B1123.

1. $\int_\Gamma xy\,dy - y^2\,dx$



SLUTET OMRÅDE. INGA SING. PUNKTER.

GREENS FORMEL:

$$\int_\Gamma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D (y-(-2y))\,dx\,dy$$

$$= \int_0^1\int_0^1 3y\,dx\,dy = \int_0^1 \left[3yx\right]_0^1 dy = \int_0^1 3y\,dy = \left[\frac{3y^2}{2}\right]_0^1 = \frac{3}{2}.$$

ALT LÖSNING: PARAMETRISERA Γ.

$$\int_\Gamma = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4}$$

1) Γ_1: $\begin{cases} x = t \\ dx = dt \\ y = 0 \\ dy = 0 \end{cases}$ $0 \le t \le 1$ $\quad \int_{\Gamma_1} = \int_0^1 0\,dt = 0$

2) Γ_2: $\begin{matrix} x = 1 \\ dx = 0 \\ y = t \\ dy = dt \end{matrix}$ $0 \le t \le 1$ $\quad \int_{\Gamma_2} = \int_0^1 t\,dt = \frac{1}{2}$

3) Γ_3: $\begin{matrix} x = t \\ dx = dt \\ y = 1 \\ dy = 0 \end{matrix}$ $0 \le t \le 1$ $\quad \int_{\Gamma_3} = \int_1^0 -1\,dt = \int_0^1 dt = 1$

4) Γ_4: $\begin{matrix} x = 0 \\ dx = 0 \\ y = t \\ dy = dt \end{matrix}$ $0 \le t \le 1$ $\quad \int_{\Gamma_4} = \int_1^0 0 = 0$

$\int_\Gamma = \frac{1}{2} + 1 = \frac{3}{2}$ $\qquad$ <u>SVAR</u>: $\frac{3}{2}$

2.

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \det A = \cos^2\theta + \sin^2\theta = 1$$

A ÄR INVERTERBAR OM $\det A \neq 0$.

$\det A = 1 \Rightarrow$ A HAR INVERS FÖR ALLA θ.

SÖK A^{-1}

$$\left(\begin{array}{ccc|ccc} \cos\theta & \sin\theta & 0 & 1 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)\begin{array}{l}\cdot\sin\theta \\ \cdot\cos\theta \\ \\ \end{array} \sim \left(\begin{array}{ccc|ccc} \cos\theta\sin\theta & \sin^2\theta & 0 & \sin\theta & 0 & 0 \\ -\cos\theta\sin\theta & \cos^2\theta & 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} \cos\theta\sin\theta & \sin^2\theta & 0 & \sin\theta & 0 & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) (-\sin^2\theta) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} \cos\theta\sin\theta & 0 & 0 & \overbrace{\sin\theta - \sin^3\theta}^{\sin\theta(1-\sin^2\theta)} & -\cos\theta\sin^2\theta & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} \cos\theta\sin\theta & 0 & 0 & \sin\theta\cos^2\theta & -\cos\theta\sin^2\theta & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \cdot \frac{1}{\cos\theta\sin\theta}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

ALT: A ÄR EN ON-MATRIS $\Rightarrow A^{-1} = A^T$.

<u>SVAR</u>: A INV. BAR FÖR ALLA θ. $A^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3.

$$\ell: \begin{cases} x = -1 + 3t \\ y = 5 + 2t \\ z = 2 - t \end{cases} \quad P_0 = (-1,5,2) \quad \bar{v}_\ell = (3,2,-1) \qquad \Pi: 2x - 4y + 2z = 9 \quad \bar{n}_\Pi: (2,-4,2) = 2(1,-2,1)$$

DET SÖKTA PLANETS NORMAL $\bar{n}_s$ ÄR VINKELRÄT MOT $\bar{v}_\ell$ OCH $\bar{n}_\Pi$.

$$\bar{n}_s = \bar{v}_\ell \times \bar{n}_\Pi = \begin{vmatrix} \bar{e}_x & \bar{e}_y & e_z \\ 3 & 2 & -1 \\ 1 & -2 & 1 \end{vmatrix} = (2-2, -1-3, -6-2) = (0,-4,-8) = -4(0,1,2)$$

DET SÖKTA PLANET:

$$(0,1,2)(x+1, y-5, z-2) = 0 \Leftrightarrow y - 5 + 2z - 4 = 0$$
$$y + 2z - 9 = 0$$

<u>SVAR</u>: $y + 2z - 9 = 0$

4. $$\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+2)^n = \left| x+2 = y \right| = \sum \frac{n^2}{2^n} \cdot y^n \qquad a_n = \frac{n^2}{2^n}$$

I) FÖR VILKA y KONV. SERIEN?

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\frac{n^2}{2^n}}{\frac{(n+1)^2}{2^{n+1}}} = \lim_{n\to\infty} \frac{n^2}{2^n} \cdot \frac{2^{n+1}}{(n+1)^2} =$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{n^2 \cdot 2}{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}+\frac{1}{n^2}} = \underline{\underline{2}}$$

KONV. FÖR $|y| < 2$.

II) $y = -2 \Rightarrow x+2 = -2 \Leftrightarrow x = -4$

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} \cdot (-2)^n = \sum_{n=0}^{\infty} n^2 \cdot \left(\frac{-2}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n n^2 = \sum_{n=0}^{\infty} b_n$$

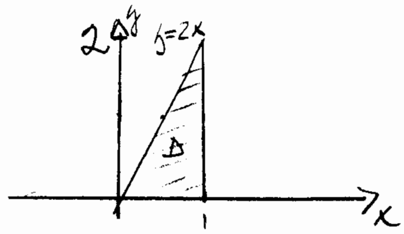
$\lim_{n\to\infty} b_n \neq 0$!! DIVERGENT.

$y = 2 \Rightarrow x = 0$

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} \cdot 2^n = \sum_{n=0}^{\infty} n^2 \quad \text{SOM ÄR DIVERGENT.}$$

<u>SVAR</u>! SERIEN KONV FÖR $-4 < x < 0$.

5. $$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy =$$



$$\int_0^1 \left(\int_0^{2x} e^{x^2}\,dy\right) dx =$$

$$= \int_0^1 \left[y e^{x^2}\right]_0^{2x} dx = \int_0^1 2x e^{x^2}\,dx = \left[e^{x^2}\right]_0^1 =$$

$$= e^1 - e^0 = e - 1.$$

<u>SVAR</u>! $e-1$

6.

$$\begin{cases} x + 2y + 4z = 2 \\ 3x + y + 6z = 3 \\ kx + 3y + 2z = 4 \end{cases} \qquad \underbrace{\begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \; ; \; A\bar{x} = \bar{b}$$

ICKE-HOMOGENT SYSTEM

$$\det A = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{vmatrix} = 2 + 12k + 36 - 4k - 18 - 12 = 8 + 8k$$

$$\det A = 0 \iff k = -1$$

1) OM $k \neq -1$ FINNS UNIK LÖSNING

CRAMERS REGEL GER:

$$x = \frac{\begin{vmatrix} 2 & 2 & 4 \\ 3 & 1 & 6 \\ 4 & 3 & 2 \end{vmatrix}}{\det A} = \frac{1 + 48 + 36 - 16 - 36 - 12}{8 + 8k} = \frac{24}{8(1+k)} = \frac{3}{1+k}$$

$$y = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 3 & 3 & 6 \\ k & 4 & 2 \end{vmatrix}}{\det A} = \frac{6 + 12k + 48 - 12k - 24 - 12}{8 + 8k} = \frac{18}{8(1+k)} = \frac{9}{4(1+k)}$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 3 \\ k & 3 & 4 \end{vmatrix}}{\det A} = \frac{4 + 6k + 18 - 2k - 9 - 24}{8 + 8k} = \frac{4k - 11}{8 + 8k}$$

2) $k = -1$ GER MED GAUSSEL.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 3 & 1 & 6 & 3 \\ -1 & 3 & 2 & 4 \end{array}\right) \begin{matrix} (-3) & (+1) \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & -5 & -6 & -3 \\ 0 & 5 & 6 & 6 \end{array}\right) (1) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & -5 & -6 & -4 \\ 0 & 0 & 0 & 3 \end{array}\right) \longleftarrow \text{ORIMLIGT} \;\; !! \;\; \text{INGA LÖSNINGAR}$$

<u>SVAR</u>: $k \neq -1$:

$$\begin{cases} x = \dfrac{3}{1+k} \\ y = \dfrac{9}{4(1+k)} \\ z = \dfrac{4k-11}{8+8k} \end{cases}$$

$k = -1$: INGA LÖSNINGAR

7. $f:(u,v) = \left(x^y, \frac{x}{y}\right) = \left(e^{y\ln x}, \frac{x}{y}\right) \quad (x,y) = (1,2)$

$$\frac{d(u,v)}{d(x,y)} = J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} yx^{y-1} & \ln x\, e^{y\ln x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix}$$

$$J_f(1,2) = \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} \qquad \det J_f = \begin{vmatrix} 2 & 0 \\ \frac{1}{2} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{2} \neq 0$$

∴ DIFFERENTIERBAR INVERS EXISTERAR.

$$\left(J_f(1,2)\right)^{-1} = \frac{d(x,y)}{d(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{1}{-\frac{1}{2}} \begin{pmatrix} -\frac{1}{4} & 0 \\ -\frac{1}{2} & 2 \end{pmatrix} =$$

$$= -2 \begin{pmatrix} -\frac{1}{4} & 0 \\ -\frac{1}{2} & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -4 \end{pmatrix}$$

SVAR: $\frac{\partial x}{\partial u} = \frac{1}{2}$, $\frac{\partial x}{\partial v} = 0$

$\frac{\partial y}{\partial u} = 1$ $\quad \frac{\partial y}{\partial v} = -4$

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8. * $2x^2 + 2y^2 + az^2 + 2axy = 1$. (EN ELLIPSOID HAR FORMELN $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$)

EN SYMMETRISK MATRIS Q ÄR!

$$Q = \begin{pmatrix} 2 & a & 0 \\ a & 2 & 0 \\ 0 & 0 & a \end{pmatrix}$$ VILKA EGENVÄRDEN HAR Q?

$$\begin{vmatrix} 2-\lambda & a & 0 \\ a & 2-\lambda & 0 \\ 0 & 0 & a-\lambda \end{vmatrix} = 0 \iff (2-\lambda)^2(a-\lambda) - a^2(a-\lambda) =$$

$$(a-\lambda)\left((2-\lambda)^2 - a^2\right) = 0$$

1) $a - \lambda = 0 \iff \lambda_1 = a$

2) $(2-\lambda)^2 - a^2 = 0$

$(2-\lambda)^2 = a^2$

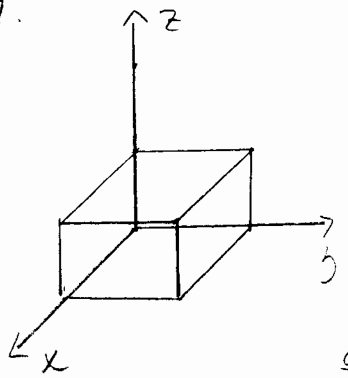
$2 - \lambda = \pm a \implies \lambda_2 = 2 - a \quad o \quad \lambda_3 = 2 + a$

FÖR ATT * SKALL VARA EN ELLIPSOID MÅSTE ALLA EGENVÄRDENA VARA > 0. VILKET GER

$a > 0$, $2 - a > 0$ OCH $2 + a > 0$. TILLSAMMANS

$0 < a < 2$. SVAR: $0 < a < 2$.

9.



$V = xyz \qquad 6x + 4y + 3z = 24$

$z = \frac{1}{3}(24 - 6x - 4y) = 8 - 2x - \frac{4}{3}y$

$V(x,y) = xy(8 - 2x - \frac{4}{3}y) = 8xy - 2x^2y - \frac{4}{3}xy^2$

SÖK MAXIMALA VÄRDET PÅ V.

$\frac{\partial V}{\partial x}: \; 8y - 4xy - \frac{4}{3}y^2 = 4y(2 - x - \frac{y}{3})$

$\frac{\partial V}{\partial y}: \; 8x - 2x^2 - \frac{8}{3}xy = 2x(4 - x - \frac{4}{3}y)$

I: $\frac{\partial V}{\partial x} = 0 \Rightarrow 4y(2 - x - \frac{y}{3}) = 0$ EJ $y = 0$, $x = 2 - \frac{y}{3}$ ($y = 0$ ÄR ORIMLIGT!)

II: $\frac{\partial V}{\partial y} = 0 \Rightarrow 2x(4 - x - \frac{4}{3}y) = 0 \Leftrightarrow x = 0$ (ORIMLIGT!)

$x = 4 - \frac{4}{3}y$

$2 - \frac{y}{3} = 4 - \frac{4}{3}y \Leftrightarrow y = 2 \Rightarrow x = 2 - \frac{2}{3} = \frac{4}{3}$

!! STATIONÄRA PUNKTER ÄR $(0,0)$ o $(\frac{4}{3}, 2)$

EFTERSOM $(0,0)$ GER $V(0,0) = 0$ (DET BLIR INGET RÄTBLOCK)

ÄR $V(\frac{4}{3}, 2) = \frac{8}{3}(8 - \frac{8}{3} - \frac{8}{3}) = \frac{8}{3}(8 - \frac{16}{3}) = \frac{8}{3}(\frac{24-16}{3}) = \frac{8}{3} \cdot \frac{8}{3} = \frac{64}{9}$

DEN MAXIMALA VOLYMEN.

$\left(\frac{\partial^2 V}{\partial x^2} = -4y \qquad \frac{\partial^2 V}{\partial x \partial y} = 8 - 4x - \frac{8}{3}y \qquad \frac{\partial^2 V}{\partial y^2} = -\frac{8}{3}x \right.$

$\left. A = -8 \qquad B = 8 - \frac{16}{3} - \frac{16}{3} = \frac{24-32}{3} = -\frac{8}{3} \qquad C = -\frac{32}{9} \qquad AC - B^2 = \frac{64}{3} > 0 \right)$

<u>SVAR</u>: MAXIMALA VOLYMEN ÄR $\frac{64}{9}$ kubikenheter

10.

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) UNDERSÖK OM $\lim_{(x,y) \to (0,0)} f(x,y) = 0$

$\lim_{(x,x^2) \to (0,0)} f(x, x^2) = \frac{3x^4}{x^4 + x^4} = \frac{3x^4}{2x^4} = \frac{3}{2}$

$\lim_{(x,-x^2) \to (0,0)} f(x, -x^2) = \frac{-3x^4}{x^4 + x^4} = \frac{-3x^4}{2x^4} = -\frac{3}{2}$

OLIKA !! GRÄNSVÄRDET EXISTERAR EJ $\Rightarrow$

$f(x,y)$ ÄR EJ KONTINUERLIG I $(0,0)$

10b) $f'_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} =$

$= \lim_{h \to 0} \frac{\frac{0}{h^2} - 0}{h} = 0$

$f'_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} =$

$= \lim_{k \to 0} \frac{\frac{0}{k^2} - 0}{k} = 0$ <u>SVAR</u>: $f'_x(0,0) = f'_y(0,0) = 0$

10c) f ÄR <u>EJ</u> DIFFERENTIERBAR I $(0,0)$

EFTERSOM f EJ ÄR KONTINUERLIG I $(0,0)$.