FÖRSLAG TILL LÖSNING TEN 5B1133
MATEMATIK II FÖR LÄRARE. 2005-01-15

1. $F(x,y,z) = xy + \frac{x}{z} + \frac{2z}{y}$ $P_0 = (-1,1,-1)$

$\text{GRAD } F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(y + \frac{1}{z},\ x - \frac{2z}{y^2},\ -\frac{x}{z^2} + \frac{2}{y}\right)$

$\text{GRAD } F(-1,1,-1) = \left(1 + \frac{1}{-1},\ -1 - \frac{2(-1)}{1},\ -\frac{(-1)}{1} + \frac{2}{1}\right) = (0,1,3)$

$\bar{v} = (7,4,4)$ $|\bar{v}| = \sqrt{7^2+4^2+4^2} = \sqrt{49+16+16} = \sqrt{81} = 9$

$\bar{e} = \frac{\bar{v}}{|\bar{v}|} = \frac{1}{9}(7,4,4)$

$\frac{dF}{d\bar{e}} = \text{GRAD } F \cdot \bar{e} = (0,1,3) \cdot \frac{1}{9}(7,4,4) = \frac{1}{9}(0\cdot 7 + 1\cdot 4 + 3\cdot 4) =$

$= \frac{1}{9}(4+12) = \frac{16}{9}$. MAXIMAL $\frac{dF}{d\bar{e}}$ ANTAS I GRAD F. RIKTN.

$\left(\frac{dF}{d\bar{v}}(-1,1,1)\right)_{max} = \text{grad } F \cdot \frac{\text{grad } F}{|\text{grad } F|} = |\text{grad } F| = \sqrt{1+3^2} = \sqrt{10}$

SVAR: $\frac{dF}{d\bar{e}} = \frac{16}{9}$, $\left(\frac{dF}{d\bar{v}}\right)_{max} = \sqrt{10}$

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2. $\begin{cases} x = t \\ y = t^2 \\ z = \frac{2}{3}t^3 \end{cases}$ $\begin{cases} \dot{x} = 1 \\ \dot{y} = 2t \\ \dot{z} = 2t^2 \end{cases}$ $P_0 = (1,1,\frac{2}{3})$

TANGENTEN: $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ $t = 1$ i P_0
$\bar{v} = (a,b,c) = (\dot{x},\dot{y},\dot{z})(P_0)$
$= (1,\ 2\cdot 1,\ 2\cdot 1^2) = (1,2,2)$

$\begin{cases} x = 1 + t \\ y = 1 + 2t \\ z = \frac{2}{3} + 2t \end{cases}$ ÄR TANGENTEN.

$L = \int_{t_0}^{t_1} \sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2}\, dt = \int_0^1 \sqrt{1^2 + (2t)^2 + (2t^2)^2}\, dt =$

$= \int_0^1 \sqrt{1 + 4t^2 + 4t^4}\, dt = \int_0^1 \sqrt{(1+2t^2)^2}\, dt = \int_0^1 |1 + 2t^2|\, dt$

$= \int_0^1 (1 + 2t^2)\, dt = \left[t + \frac{2}{3}t^3\right]_0^1 = 1 + \frac{2}{3} = \frac{5}{3}$ SVAR: $\frac{5}{3}$ l.e

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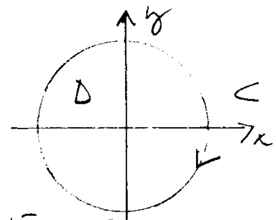
3. $I = \int_C (3x^2 + 2xy)\,dx + (y + 2x + x^2)\,dy$



SLUTET OMRÅDE. INGA SING. PKT

$\Rightarrow$ GREENS FORMEL KAN ANVÄNDAS. NEG. RIKTN

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial Q}{\partial x} = 2 + 2x \qquad \frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x} = 2 + 2x - 2x = 2$$

$$I = \underset{\substack{\uparrow \\ \text{neg. rikt } D}}{-} \iint 2\,dx\,dy = -2 \cdot \underset{\substack{\uparrow \\ \text{arean av cirkeln}}}{\pi \cdot 1^2} = -2\pi.$$

SVAR: -2π

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4. $P_0 = (1, -1, 1) \qquad (x, y, z) = (1, 0, 1) + t(6, 3, 1)$

TVÅ PUNKTER I PLANET ÄR T. EX.

$P_1 = (1, 0, 1)$ OCH $P_2 = (7, 3, 2)$. TVÅ VEKTORER I PLANET ÄR:

$\overrightarrow{P_0P_1} = (1, 0, 1) - (1, -1, 1) = (0, 1, 0)$

$\overrightarrow{P_0P_2} = (7, 3, 2) - (1, -1, 1) = (6, 4, 1)$

$$\bar{n}_\pi = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & 1 & 0 \\ 6 & 4 & 1 \end{vmatrix} = (1-0,\ 0-0,\ 0-6) = (1, 0, -6)$$

PLANET π: $\bar{n} \cdot (\bar{r} - \bar{r}_0) = (1, 0, -6) \cdot (x-1, y+1, z-1) = 0$

π: $x - 1 - 6(z-1) = 0$

$x - 6z + 5 = 0$

SVAR: $x - 6z + 5 = 0$

UPPGIFT 5 ~~PÅ~~ SID 5.

6. $$\iiint_V \frac{\sqrt{z}}{1 + x^2 + y^2}\,dx\,dy\,dz \qquad x^2 + y^2 \le 1,\ 0 \le z \le x^2 + y^2$$

$$\iint_{x^2+y^2\le 1} \left(\int_0^{x^2+y^2} \frac{\sqrt{z}}{1+x^2+y^2}\,dz \right) dx\,dy = \iint_{x^2+y^2\le 1} \left[\frac{\frac{2}{3} z^{\frac{3}{2}}}{1+x^2+y^2} \right]_0^{x^2+y^2} dx\,dy =$$

$$= \frac{2}{3} \iint_{x^2+y^2\le 1} \frac{(x^2+y^2)^{3/2}}{1+x^2+y^2}\,dx\,dy = \left| \begin{matrix} \text{POLÄRA KOORD} \\ dx\,dy = r\,dr\,d\theta \\ 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{matrix} \right| = \frac{2}{3} \int_0^{2\pi} \int_0^1 \frac{(r^2)^{3/2} \cdot r}{1 + r^2}\,dr\,d\theta$$

FORTS

6. FORTS.

$$= \frac{2}{3}\int_0^{2\pi}\int_0^1 \frac{r^4}{1+r^2}\,dr\,d\theta = \frac{2}{3}\int_0^{2\pi} d\theta \int_0^1 \left(r^2-1+\frac{1}{1+r^2}\right)dr =$$

$$= \frac{2}{3}\cdot 2\pi \cdot \left[\frac{r^3}{3} - r + \arctan r\right]_0^1 = \frac{4\pi}{3}\left(\frac{1}{3}-1+\frac{\pi}{4}\right) =$$

$$= \frac{4\pi}{3}\left(-\frac{2}{3}+\frac{\pi}{4}\right) = \frac{\pi^2}{3} - \frac{8\pi}{9}$$

SVAR: $\frac{\pi^2}{3} - \frac{8\pi}{9}$

7. $i(z^2+z) = z-1-2i$

$$z^2+z = \frac{1}{i}(z-1-2i) = \frac{i}{i^2}(z-1-2i) = -i(z-1-2i)$$

$$= -iz + i + 2i^2 = -iz + i - 2$$

$$z^2 + \underbrace{z + iz}_{z(1+i)} - i + 2 = 0$$

KVADRATKOMPLETTERA!

$$\left(z+\frac{1+i}{2}\right)^2 - \frac{(1+i)^2}{2^2} - i + 2 = 0$$

$$\left(z+\frac{1+i}{2}\right)^2 = \frac{(1+i)^2}{4} + i - 2$$

$$\left(z+\frac{1+i}{2}\right)^2 = \frac{2i}{4} + i - 2 = -2 + \frac{3}{2}i$$

$$z + \frac{1+i}{2} = \pm\sqrt{-2+\frac{3}{2}i}$$

SÄTT $a + ib = \sqrt{-2+\frac{3}{2}i}$

KVADRERA $a^2 - b^2 + 2abi = -2 + \frac{3}{2}i$

$$\begin{matrix} \mathrm{I} \\ \mathrm{II} \end{matrix}\begin{cases} a^2 - b^2 = -2 \\ 2ab = \frac{3}{2} \end{cases}$$

$b = \frac{3}{4a}$ INSÄTT I I

$$a^2 - \frac{9}{16a^2} = -2$$

$$16a^4 + 32a^2 - 9 = 0$$

$$a^4 + 2a^2 - \frac{9}{16} = 0$$

$$a^2 = -1 \pm \sqrt{1+\frac{9}{16}}$$

$$a^2 = -1 \pm \sqrt{\frac{25}{16}} = -1 \pm \frac{5}{4}$$

$$a^2 = \frac{1}{4}$$

$\left(a^2 = -\frac{9}{4}\right.$ ej def $\left.\right)$

$$a = \pm\frac{1}{2} \Rightarrow b = \pm\frac{3}{2}$$

$$z + \frac{1+i}{2} = \pm\left(\frac{1}{2}+\frac{3}{2}i\right)$$

$$z = -\left(\frac{1+i}{2}\right) + \left(\frac{1}{2}+\frac{3}{2}i\right) = i$$

$$z = -\left(\frac{1+i}{2}\right) - \left(\frac{1}{2}+\frac{3}{2}i\right) = -1-2i$$

SVAR: $z = i$
$z = -1-2i$

8.

$$\begin{cases} u = x+2y \\ v = 2x-y \end{cases} \qquad f(x,y) = g(u,v) = g(x+2y,\, 2x-y)$$

$$f'_x = \frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial g}{\partial u}\cdot 1 + \frac{\partial g}{\partial v}\cdot 2$$

$$f'_y = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial g}{\partial u}\cdot 2 + \frac{\partial g}{\partial v}\cdot -1$$

$$f''_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \left(\frac{\partial}{\partial u} + 2\frac{\partial}{\partial v}\right)\left(2\frac{\partial g}{\partial u} - \frac{\partial g}{\partial v}\right)$$

$$= 2\frac{\partial^2 g}{\partial u^2} - \frac{\partial^2 g}{\partial u \partial v} + 4\frac{\partial^2 g}{\partial v \partial u} - 2\frac{\partial^2 g}{\partial v^2} =$$

$$= 2\frac{\partial^2 g}{\partial u^2} + 3\frac{\partial^2 g}{\partial u \partial v} - 2\frac{\partial^2 g}{\partial v^2} = \text{SVAR}$$

9.

$$u(x,y) = e^{x-y}(2+xy) = e^x \cdot e^{-y}(2+xy)$$

$$u'_x = e^{x-y}(2+xy) + e^{x-y}\cdot y = e^{x-y}(2+xy+y)$$

$$u'_y = -e^{x-y}(2+xy) + e^{x-y}\cdot x = e^{x-y}(-2-xy+x)$$

$u'_x = 0 \iff 2+xy+y = 0$ I $\qquad$ I + II GER $x+y=0$

$u'_y = 0 \iff -2-xy+x = 0$ II $\qquad \iff y = -x$

$y=-x$ INSATT I I GER $2-x^2-x=0$

$x^2+x-2=0$; $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+\frac{8}{4}} = -\frac{1}{2} \pm \frac{3}{2}$ $\qquad x=1$, $x=-2$

STATIONÄRA PUNKTER : $(1,-1)$ OCH $(-2,2)$

$$u''_{xx} = e^{x-y}(2+xy+y) + e^{x-y}\cdot y = e^{x-y}(2+xy+2y)$$

$$u''_{xy} = -e^{x-y}(2+xy+y) + e^{x-y}(x+1) = e^{x-y}(-1-xy-y+x)$$

$$u''_{yy} = -e^{x-y}(-2-xy+x) + e^{x-y}(-x) = e^{x-y}(2+xy-2x)$$

	$(1,-1)$	$(-2,2)$
A	$e^2\cdot(-1)$	$e^{-4}\cdot 2$
B	$e^2\cdot 2$	$e^{-4}\cdot(-1)$
C	$e^2\cdot -1$	$e^{-4}\cdot 2$
$AC-B^2$	$e^2(1-2^2)<0$	$e^{-4}(4-1)>0$

$(1,-1)$ ÄR EN SADELPUNKT

$(-2,2)$ ÄR EN LOK. MINPUNKT

SVAR: $u(1,-1) = e^2$ SADELP.

$u(-2,2) = -2e^{-4}$ LOK. MIN

5. $\sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{1}{n^2}} = \sum_{n=1}^{\infty} a_n$ VÄLJ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{n} e^{-\frac{1}{n^2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}\left(1 - \frac{1}{n^2} + \frac{1}{2}\left(-\frac{1}{n^2}\right)^2 + O\left(\frac{1}{n^6}\right)\right)}{\frac{1}{n}} = 1$$

$\sum \frac{1}{n}$ ÄR DIV. $\Rightarrow$ $\sum \frac{1}{n} e^{-\frac{1}{n^2}}$ ÄR DIV ENLIGT JÄMFÖRELSEPRINCIPEN. <u>SVAR</u>: DIVERGENT.

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10. $$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}$$

$$u = u(x,y) = u(x(u,v), y(u,v))$$
$$v = v(x,y) = v(x(u,v), y(u,v))$$

DERIVERA DESSA UTTRYCK

$$\frac{\partial}{\partial u}: \quad \left.\begin{aligned} 1 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} \\ 0 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} \end{aligned}\right\}.$$

LÖS UT $\frac{\partial x}{\partial u}$ OCH $\frac{\partial y}{\partial u}$ MED T. EX CRAMERS REGEL

$$\frac{\partial x}{\partial u} = \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}} = \frac{\frac{\partial v}{\partial y}}{J} \quad \text{OCH} \quad \frac{\partial y}{\partial u} = \frac{-\frac{\partial v}{\partial x}}{J}$$

$$\frac{\partial}{\partial v}: \quad \left.\begin{aligned} 0 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ 1 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{aligned}\right\} \Rightarrow \frac{\partial x}{\partial v} = -\frac{\frac{\partial u}{\partial y}}{J}$$

OCH $\frac{\partial y}{\partial v} = \frac{\frac{\partial u}{\partial x}}{J}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v} = \frac{\frac{\partial v}{\partial y}}{J}\cdot\frac{\frac{\partial u}{\partial x}}{J} - \frac{\left(-\frac{\partial v}{\partial x}\right)}{J}\cdot\frac{\left(-\frac{\partial u}{\partial y}\right)}{J} =$$
$$= \frac{\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}}{J^2} = \frac{J}{J^2} = \frac{1}{J} \quad \text{V.S.V}$$