# FÖRSLAG TILL LÖSNING. TENTAMEN MATEMATIK II

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1a) $P=(1,2,3)$ $\quad \bar{v}=(2,-4,-1)$

LINJEN ÄR $L: \begin{cases} x = 1 + 2t \\ y = 2 - 4t \\ z = 3 - t \end{cases}$

LIGGER L I PLANET $\pi: x-2y+3z=6$.

PLANETS NORMALVEKTOR ÄR $\bar{n}_\pi \ (1,-2,3)$

$\bar{n}_\pi \cdot \bar{v} = (1,-2,3)\cdot(2,-4,-1) = 2+8-3 \neq 0$

$\therefore$ PLANET OCH LINJEN ÄR INTE PARALLELLA OCH LINJEN KAN EJ LIGGA I PLANET.

(ALT. SÄTT IN LINJEN I PLANET:

$1+2t-2(2-4t)+3(3-t)=6$

$1+2t-4+8t+9-3t=6$

$7t=0$, $t=0$ $\quad \therefore$ LINJEN SKÄR $\pi$ I PUNKT. $(1,2,3)$ )

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1b)

i) $A=\begin{pmatrix}1&2&0\\0&2&b\\1&3&1\end{pmatrix}$ $\quad \det A = 2+2b-3b \neq 0 \iff b\neq 2.$

ii) VÄLJ T EX $b=1$ $\quad A=\begin{pmatrix}1&2&0\\0&2&1\\1&3&1\end{pmatrix}$ $\quad \det A = 1$

$$\left(\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&2&1&0&1&0\\1&3&1&0&0&1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&2&1&0&1&0\\0&1&1&-1&0&1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&1&0&1&1&-1\\0&1&1&-1&0&1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1&0&0&-1&-2&2\\0&1&0&1&1&-1\\0&0&1&-2&-1&2\end{array}\right) \qquad A^{-1}=\begin{pmatrix}-1&-2&2\\1&1&-1\\-2&-1&2\end{pmatrix}$$

SVAR: $b\neq 2$. OM $b=1$ ÄR $A^{-1}=\begin{pmatrix}-1&-2&2\\1&1&-1\\-2&-1&2\end{pmatrix}$

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2a) $(1,2,1)$, $(1,0,2)$ O $(1,1,0)$

$$\begin{vmatrix}1&1&1\\2&0&1\\1&2&0\end{vmatrix} = 0+1+4-0-2-0 = 5-2 = 3 \neq 0$$

$\therefore$ LINJÄRT OBEROENDE.

SVAR: LINJÄRT OBEROENDE

b) $q = 2x^2 + 4y^2 + 6yz - 4z^2$

$K = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{vmatrix}$ SÖK EGENVÄRDEN:

KAR. EKV $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 3 \\ 0 & 3 & -4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(4-\lambda)(-4-\lambda) - 9(2-\lambda) = 0$

$(2-\lambda)\left((4-\lambda)(-4-\lambda) - 9\right) = 0 \qquad 2-\lambda = 0 \Leftrightarrow \lambda = 2$

$-16 - 4\lambda + 4\lambda + \lambda^2 - 9 = 0$

$\lambda^2 - 25 = 0$

$\lambda = \pm 5$

!! EGENVÄRDEN ÄR $-5$, $2$ OCH $5$.

HUVUDAXELFORM BLIR $h = -5u^2 + 2v^2 + 5w^2$.

SVAR: $h = -5u^2 + 2v^2 + 5w^2$

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c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n+2}{4n^2-3} = \frac{5}{1} - \frac{8}{13} + \frac{11}{33} \qquad = \sum_{n=1}^{\infty} a_n$

1) ALT. SERIE

2) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3n+2}{4n^2-3} = \lim_{n\to\infty} \frac{n^2\left(\frac{3}{n} + \frac{2}{n^2}\right)}{n^2\left(4 - \frac{3}{n^2}\right)} =$

$= \lim_{n\to\infty} \frac{\frac{3}{n} + \frac{2}{n^2}}{4 - \frac{3}{n^2}} = \frac{0+0}{4} = 0$

3) $|a_n|$ AVTAGANDE ?

$f(x) = \frac{3x+2}{4x^2-3} \qquad f'(x) = \frac{3(4x^2-3) - (3x+2)\cdot 8x}{(4x^2-3)^2} =$

$= \frac{12x^2 - 9 - 24x^2 - 16x}{(4x^2-3)^2} = \frac{-12x^2 - 16x - 9}{(4x^2-3)^2} = \frac{-(12x^2 + 16x + 9)}{(4x^2-3)^2} < 0$ DÅ $x \geq 1$.

!! $|a_n|$ ÄR AVTAGANDE.

OCH SERIEN ÄR LEIBNIZKONVERGENT.

SVAR: KONVERGENT

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**3b)** $f(x,y,z) = x^2yz^2 + \sin yz \qquad (2,\pi,1)$

$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(2xyz^2,\ x^2z^2 + z\cos yz,\ 2x^2yz + y\cos yz\right)$

$\operatorname{grad} f(2,\pi,1) = \left(2\cdot 2\cdot\pi\cdot 1^2,\ 2^2\cdot 1 + \cos\pi,\ 2\cdot 2^2\cdot\pi\cdot 1 + \pi\cos\pi\right) =$

$= (4\pi,\ 4-1,\ 8\pi - \pi) = (4\pi, 3, 7\pi)$ <u>SVAR</u>: $(4\pi, 3, 7\pi)$

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**4a)** $\frac{\partial^2 f}{\partial u^2} \qquad f(x,y) = f\left(u, v - \frac{u^2}{2}\right) \qquad \begin{cases} x = u \\ y = v - \frac{u^2}{2} \end{cases} \Rightarrow \begin{cases} u = x \\ v = \frac{x^2}{2} + y \end{cases}$

$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot(-u) = \frac{\partial f}{\partial x} - u\frac{\partial f}{\partial y}$

$\frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = |u = x| = \left(\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial y}$

$- x\frac{\partial^2 f}{\partial x\partial y} - x\frac{\partial^2 f}{\partial y\partial x} + x^2\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} - 2x\frac{\partial^2 f}{\partial x\partial y} + x^2\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial y}$

<u>SVAR</u>: $\frac{\partial^2 f}{\partial x^2} - 2x\frac{\partial^2 f}{\partial x\partial y} + x^2\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial y}$

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**4b)** $f(x,y) = e^{xy+x}$

$e^x = 1 + x + \frac{x^2}{2!} + O(x^3)$

$e^{(xy+x)} = 1 + xy + x + \frac{(xy+x)^2}{2} + O(r^3) =$

$= 1 + xy + x + \frac{x^2y^2}{2} + x^2y + \frac{x^2}{2} + O(r^3) =$

$= 1 + xy + x + \frac{x^2}{2} + O(r^3)$ <u>SVAR</u>: $1 + x + xy + \frac{x^2}{2} + O(r^3)$

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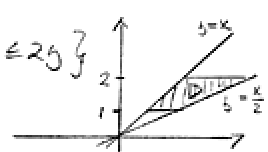
**5a)**

$\iint_D xy\,dx\,dy = \qquad D = \{(x,y) \mid 1 \le y \le 2,\ y \le x \le 2y\}$

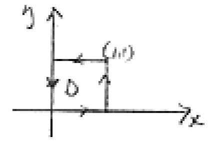



$= \int_1^2\left(\int_y^{2y} xy\,dx\right)dy = \int_1^2 \left[\frac{x^2}{2}\cdot y\right]_y^{2y} dy =$

$= \int_1^2 \left(\frac{4y^2}{2}\cdot y - \frac{y^2}{2}\cdot y\right)dy = \int_1^2 \frac{3}{2}y^3\,dy = \frac{3}{2}\left[\frac{y^4}{4}\right]_1^2 = \frac{3}{2}\left(\frac{16}{4} - \frac{1}{4}\right)$

$= \frac{3}{2}\cdot\frac{15}{4} = \frac{45}{8}$ <u>SVAR</u>: $\frac{45}{8}$

5b) $\int_\Gamma x^2y^3\,dx + 2xy\,dy$




SLUTET OMRÅDE
INGA SINGULARITETER

$P = x^2y^3 \quad Q = 2xy$

$\frac{\partial P}{\partial y} = 3x^2y^2 \quad \frac{\partial Q}{\partial x} = 2y$ GREENS FORMEL GER:

$$\int_\Gamma = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_0^1\int_0^1 (2y - 3x^2y^2)\,dx\,dy =$$

$$= \int_0^1 \left[2yx - x^3y^2\right]_0^1 dy = \int_0^1 (2y - y^2)\,dy = \left[y^2 - \frac{y^3}{3}\right]_0^1 = \frac{2}{3}.$$

(ALT. BERÄKNA ÖVER DE FYRA OLIKA RÄNDERNA)

<u>SVAR</u>: $\frac{2}{3}$.

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$f(x,y) = 3x^3 + xy^2 + 5x^2 + y^2$

$f'_x = 9x^2 + y^2 + 10x$

$f'_y = 2xy + 2y = 2y(x+1)$

$\left.\begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 9x^2 + y^2 + 10x = 0 \\ 2y(x+1) = 0 \end{array} \iff \begin{cases} x = -1 \\ y = 0 \end{cases}$ SÄTTS IN I $f'_x = 0$

$x = -1 \Rightarrow 9 + y^2 - 10 = 0 \iff y^2 = 1 \iff y = \pm 1$

$y = 0 \Rightarrow 9x^2 + 10x = 0 \quad x(9x+10) = 0 \iff \begin{array}{l} x = 0 \\ x = -\frac{10}{9} \end{array}$

STATIONÄRA PUNKTER:

$(-1,-1)$, $(-1,1)$, $(0,0)$, $(-\frac{10}{9}, 0)$

$f''_{xx} = 18x + 10 = A$

$f''_{xy} = 2y = B$

$f''_{yy} = 2x + 2 = C$

| PUNKT | A | B | C | $AC-B^2$ | KAR |
|---|---|---|---|---|---|
| $(-1,-1)$ | $-8$ | $-2$ | $0$ | $-4$ | SADEL |
| $(-1,1)$ | $-8$ | $2$ | $0$ | $-4$ | SADEL |
| $(0,0)$ | $10$ | $0$ | $2$ | $20$ | LOK. MIN |
| $(-\frac{10}{9},0)$ | $-10$ | $0$ | $-\frac{2}{9}$ | $20/9$ | LOK MAX |

SVAR: $(-1,-1)$ OCH $(-1,1)$ ÄR SADELPUNKTER
$(0,0)$ LOKAL MINPUNKT
$(-\frac{10}{9},0)$ LOKAL MAXPUNKT

7. $P_0 = (1,1,-3)$ $\bar{n}_1 = (2,2,1)$, $\bar{n}_2 = (-1,2,1)$

$\Pi_1$ : $(2,2,1)(x-1,\ y-1,\ z+3) = 0$

$2(x-1) + 2(y-1) + z + 3 = 0$

$2x + 2y + z - 2 - 2 + 3 = 0$

$2x + 2y + z = 1$

$\Pi_2$ : $(-1,2,1)(x-1,\ y-1,\ z+3) = 0$

$-(x-1) + 2(y-1) + z + 3) = 0$

$-x + 2y + z + 1 - 2 + 3 = 0$

$x - 2y - z = 2$

SKÄRNINGSLINJE : $\begin{matrix} \text{I} \\ \text{II} \end{matrix} \begin{cases} 2x + 2y + z = 1 \\ x - 2y - z = 2 \end{cases}$

I + II : $3x = 3 \iff x = 1$ INSATT I I $2y + z = -1$
$z = -1 - 2y$

LINJEN : $\begin{cases} x = 1 \\ y = t \\ z = -1 - 2t \end{cases}$ SVAR: $(1,0,-1) + t(0,1,-2)$

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8. $\int_\Gamma (ay + 2xy^2)\,dx + (bx^2y + x + 1)\,dy$ $\Gamma : (0,0) \to (2,1)$

$P = ay + 2xy^2$ $Q = bx^2y + x + 1$

$\frac{\partial P}{\partial y} = a + 4xy$ $\frac{\partial Q}{\partial x} = 2bxy + 1$. OM $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ÄR

INTEGRALEN OBEROENDE AV VÄGEN.

$a + 4xy = 2bxy + 1$ OM $a = 1$ OCH $b = 2$ ÄR $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\int_\Gamma (y + 2xy^2)\,dx + (2x^2y + x + 1)\,dy$

$P = \frac{\partial F}{\partial x} = y + 2xy^2 \Rightarrow F(x,y) = xy + x^2y^2 + \varphi(y)$

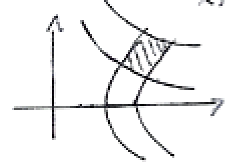
$\frac{\partial F}{\partial y} = x + 2x^2y + \varphi'(y) = Q = 2x^2y + x + 1$ VILKET GER

$\varphi'(y) = 1 \Rightarrow \varphi(y) = y + C$ $\therefore F(x,y) = xy + x^2y^2 + y + C$.

$\int_\Gamma = F(2,1) - F(0,0) = 2 + 4 + 1 + C - C = 7.$ SVAR: $a = 1$, $b = 2$, $\int = 7$

$$\iint_D x^4-y^4\,dxdy \qquad D=\left\{(x,y) \mid 1\le x^2-y^2\le 2,\ 1\le xy\le 4,\ x>0,\ y>0\right\}$$

SÄTT $\begin{cases} x^2-y^2=u \\ xy=v \end{cases}$ $\quad 1\le u\le 2$, $\quad 1\le v\le 4$



$$\frac{d(x,y)}{d(u,v)}=\frac{1}{\frac{d(u,v)}{d(x,y)}}=\frac{1}{\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}}=\frac{1}{\begin{vmatrix}2x & -2y\\ y & x\end{vmatrix}}=\frac{1}{2x^2+2y^2}=\frac{1}{2(x^2+y^2)}$$

$$\frac{d(x,y)}{d(u,v)}=\frac{1}{2(x^2+y^2)} \Rightarrow (x^2+y^2)\,dxdy=\frac{1}{2}\,dudv$$

$$\iint_D (x^4-y^4)\,dxdy=\iint_D (x^2-y^2)(x^2+y^2)\,dxdy=$$

$$=\int_1^4\int_1^2 u\cdot\frac{1}{2}\,dudv=\int_1^4\left[\frac{u^2}{4}\right]_1^2 dv=\int_1^4\left(1-\frac{1}{4}\right)dv=\left[\frac{3}{4}v\right]_1^4=$$

$$=3-\frac{3}{4}=\frac{9}{4}.$$

<u>SVAR</u>: $\frac{9}{4}$

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SE PETERMANN SID 212

EN VEKTOR $\bar{v}$ KALLAS EGENVEKTOR TILL EN MATRIS A OM DEN UPPFYLLER VILLKORET $A\bar{v}=\lambda\bar{v}$ $(\bar{v}\neq\bar{0})$ FÖR NÅGON KONSTANT $\lambda$. $\lambda$ KALLAS EGENVÄRDE TILL MATRISEN.

$A\bar{v}=\lambda\bar{v} \quad \bar{v}\neq\bar{0}$

$A\bar{v}-\lambda\bar{v}=\bar{0} \iff (A-\lambda E)\bar{v}=\bar{0}$ DETTA ÄR ETT KVAD. LINJÄRT HOMOGENT EKV. SYSTEM

ICKE-TRIVIALA LÖSNINGAR FINNS OMM DETERMINANTEN DVS $DET(A-\lambda E)=0$. (KAR. EKV)

TEX FÖR EN 2x2 MATRIS $\begin{vmatrix} a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}=0$. HÄRUR ERHÅLLS $\lambda_1$ OCH $\lambda_2$, VILKA DÄREFTER GER $\bar{v}_1$ OCH $\bar{v}_2$.

ANVÄNDN. OMRÅDEN! TRANSFORMERING TILL HUVUDAXELFORM, EXTREMVÄRDESBESTÄMNINGAR