1a)

$$\begin{cases} x + y + z = 1 \\ x + 2y + 3z = 1 \\ x + 3y + az = b \end{cases} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix} \Leftrightarrow A\bar{x} = \bar{v}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & a \end{pmatrix} \qquad \det A = 2a + 3 + 3 - 2 - 9 - a = a - 5$$

SYSTEMET SAKNAR ENTYDIG LÖSNING DÅ $\det A = 0 \Rightarrow a - 5 = 0 \Leftrightarrow a = 5$

SVAR! $a = 5$

1b)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & b \end{array}\right) (-1) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & b-1 \end{array}\right) (-2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & b-1 \end{array}\right)$$

OM $b - 1 = 0$ HAR SYSTEMET OÄNDLIGT MÅNGA LÖSNINGAR. DVS DÅ $b = 1$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right) (-1) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\begin{cases} x = 1 + z \\ y = -2z \\ z = z \end{cases} \Rightarrow \begin{cases} x = 1 + t \\ y = -2t \\ z = t \end{cases}$$

SVAR: $b = 1$ GER LÖSN.

$$\begin{cases} x = 1 + t \\ y = -2t \\ z = t \end{cases}$$

2a) $(1,2,1,0)$, $(a,a,1,0)$, $(2,1,2,1)$ OCH $(1,1,1,0)$. BAS I R^4?

UNDERSÖK DETERMINANTEN

$$\begin{vmatrix} 1 & a & 2 & 1 \\ 2 & a & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \neq 0 \qquad \begin{vmatrix} 1 & a & 2 & 1 \\ 2 & a & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)^{4+3}\begin{vmatrix} 1 & a & 1 \\ 2 & a & 1 \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= -(a + a + 2 - a - 1 - 2a) = -(a+1) \qquad a + 1 = 0 \Leftrightarrow a = -1$$

DETTA INNEBÄR ATT VEKTORERNA BILDAR EN BAS OM OCH ENDAST OM $a \neq -1$.

SVAR! $a \neq -1$

2b) $A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$ KAR. EKV: $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 6 = 0$

$$12 - 3\lambda - 4\lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$\lambda = 1$$
$$\lambda = 6$$

$\lambda_1 = 1 \Rightarrow \left(\begin{array}{cc|c} 2 & 6 & 0 \\ 1 & 3 & 0 \end{array}\right) \Leftrightarrow (1, 3 \mid 0)$ $x + 3y = 0$, $x = -3y$

$v_1 = t\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\lambda_2 = 6 \Rightarrow \left(\begin{array}{cc|c} -3 & 6 & 0 \\ 1 & -2 & 0 \end{array}\right) \Leftrightarrow (1\ -2 \mid 0)$ $x - 2y = 0$, $x = 2y$

$\bar{v}_2 = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

SVAR! $\lambda = 1$ GER $\bar{v}_1 = t\begin{pmatrix} -3 \\ 1 \end{pmatrix}$
$\lambda = 6$ GER $\bar{v}_2 = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

3a) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ $\quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}} =$

$$= \lim_{n\to\infty} \frac{2\cdot 2^n+5}{3\cdot 3^n}\cdot\frac{3^n}{2^n+5} = \lim_{n\to\infty} \frac{2\cdot 2^n+5}{3(2^n+5)} =$$

$$= \lim_{n\to\infty} \frac{2^n(2+\frac{5}{2^n})}{2^n(3+\frac{15}{2^n})} = \lim_{n\to\infty} \frac{2+\frac{5}{2^n}}{3+\frac{15}{2^n}} = \frac{2}{3} < 1$$

$\therefore$ $\sum \frac{2^n+5}{3^n}$ KONV. ENL KVOTKRIT.

SVAR: KONV.

3b) $z = f(u,v)$ $\quad \begin{cases} u = 3x+4y \\ v = xy \end{cases}$

$$z'_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = f'_u\cdot 3 + f'_v\cdot y$$

$$z'_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = f'_u\cdot 4 + f'_v\cdot x$$

$$x z'_x + y z'_y = x(3f'_u + y f'_v) + y(4f'_u + x f'_v) =$$

$$= (3x+4y)f'_u + (xy+xy)f'_v = u f'_u + 2v f'_v.$$

SVAR: $u f'_u + 2v f'_v$

4a) $\ln\left(\frac{x}{y-z}\right) = 0$ $\quad F(x,y,z) = \ln\left(\frac{x}{y-z}\right) = \ln x - \ln(y-z)$

$$\bar{n} = \text{GRAD } F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(\frac{1}{x}, -\frac{1}{y-z}, \frac{1}{y-z}\right)$$

$P_0 = (1,4,3)$ $\quad \bar{n}(1,4,3) = (1, -1, 1)$

NORMALEN: $\begin{cases} x = 1+t \\ y = 4-t \\ z = 3+t \end{cases}$ $\quad$ SVAR: $\begin{cases} x = 1+t \\ y = 4-t \\ z = 3+t \end{cases}$

4b) $z(x,y) = x^3 + y^3 - 9xy + 27$

$$\left.\begin{aligned} z'_x &= 3x^2 - 9y = 3(x^2-3y) \quad & z'_x = 0 \Rightarrow y = \frac{x^2}{3} \\ z'_y &= 3y^2 - 9x = 3(y^2-3x) \quad & z'_y = 0 \Rightarrow y^2 - 3x = 0 \end{aligned}\right\} \begin{matrix} \text{I} \\ \text{II} \end{matrix}$$

I INSATT I II: $\left(\frac{x^4}{9} - 3x\right) = 0 \iff x\left(\frac{x^3}{9} - 3\right) = 0$,

$x = 0$, $x = 3$ $\quad x = 0 \Rightarrow y = 0$, $x = 3 \Rightarrow y = 3$

FORTS

4b) FORTS.

STATIONÄRA PUNKTER ÄR (0,0) OCH (3,3)

$z''_{xx} = 6x$

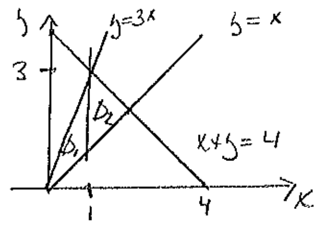
$z''_{xy} = -9$

$z''_{yy} = 6y$

	A	B	C	$AC-B^2$	KARAK.
(0,0)	0	−9	0	−81	SADELP.
(3,3)	18	−9	18	$18^2-(9)^2>0$	LOK. MIN

SVAR: $z(0,0)=27$ ÄR SADELPUNKT
$z(3,3)=0$ ÄR LOK. MIN

5a) $\iint_D (x+y-2)\,dx\,dy =$



$= \iint_{D_1} + \iint_{D_2} =$

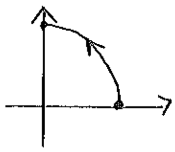
$= \int_0^1 \left(\int_x^{3x} (x+y-2)\,dy \right) dx + \int_1^2 \left(\int_x^{4-x} (x+y-2)\,dy \right) dx = \int_0^1 \left[xy + \frac{y^2}{2} - 2y \right]_x^{3x} dx$

$+ \int_1^2 \left[xy + \frac{y^2}{2} - 2y \right]_x^{4-x} dx = \int_0^1 \left(3x^2 + \frac{9}{2}x^2 - 6x - x^2 - \frac{x^2}{2} + 2x \right) dx$

$+ \int_1^2 \left(4x - x^2 + \frac{16-8x+x^2}{2} - 8 + 2x - x^2 - \frac{x^2}{2} + 2x \right) dx = \int_0^1 (6x^2 - 4x)\,dx + \int_1^2 (4x - 2x^2)\,dx$

$= \left[2x^3 - 2x^2 \right]_0^1 + \left[2x^2 - \frac{2x^3}{3} \right]_1^2 = 0 + \left(8 - \frac{16}{3} - 2 + \frac{2}{3} \right) = 6 - \frac{14}{3} = \frac{4}{3}.$ SVAR: $\frac{4}{3}$

5b) $\int_\Gamma xy^2\,dx + y\,dy \qquad x^2+y^2=1$



SÄTT $x = \cos\theta \qquad dx = -\sin\theta\,d\theta$
$y = \sin\theta \qquad dy = \cos\theta\,d\theta$

$\int_\Gamma xy^2\,dx + y\,dy = \int_0^{\frac{\pi}{2}} \cos\theta \cdot \sin^2\theta \cdot -\sin\theta\,d\theta + \sin\theta\cos\theta\,d\theta =$

$= \int_0^{\frac{\pi}{2}} (-\sin^3\theta + \sin\theta)\cos\theta\,d\theta = \left| \begin{array}{l} \sin\theta = t \\ \cos\theta\,d\theta = dt \\ \theta = 0 \Rightarrow t = 0 \\ \theta = \frac{\pi}{2} \Rightarrow t = 1 \end{array} \right| = \int_0^1 (-t^3 + t)\,dt =$

$= \left[-\frac{t^4}{4} + \frac{t^2}{2} \right]_0^1 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.$ SVAR: $\frac{1}{4}$

6. $P_0 = (1,-1,1)$

$L: \; x = 2y-1 = 3z-2 \iff \begin{cases} x = t \\ 2y-1 = t \\ 3z-2 = t \end{cases} \iff \begin{matrix} x = t \\ y = (1+t)/2 \\ z = (2+t)/3 \end{matrix}$

PLANET π: $\bar{n}\cdot(x-x_0,\, y-y_0, z-z_0) = 0$

SÖK $\bar{n}$.

TVÅ PUNKTER PÅ L ÄR $P_1 = (0, 1/2, 2/3)$ OCH $P_2 = (1,1,1)$

$\bar{v}_1 = \overrightarrow{P_1P_0} = (0,1/2,2/3) - (1,-1,1) = (-1, 3/2, -1/3)$

$\bar{v}_2 = \overrightarrow{P_2P_0} = (1,1,1) - (1,-1,1) = (0,2,0)$

$\bar{n} = \bar{v}_1 \times \bar{v}_2 = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ -1 & 3/2 & -1/3 \\ 0 & 2 & 0 \end{vmatrix} = (0+2/3, 0, -2) = (\frac{2}{3}, 0, -2) = \frac{2}{3}(1,0,-3)$

$\therefore \pi: \; (1,0,-3)(x-1,\, y+1,\, z-1) = 0 \iff (x-1)-3(z-1) = 0$

$x-1-3z+3 = 0 \iff x-3z+2 = 0$ <u>SVAR</u>: $x-3z+2 = 0$

7. $\bar{a}(t) = (1, \frac{1}{\sqrt{t}}) \Rightarrow \bar{v}(t) = (t+a,\, 2\sqrt{t}+b)$ a, b konst.

$\bar{v}(1) = (0,2)$ INSATT I $\bar{v}(t)$ GER $1+a = 0 \iff a = -1$

OCH $2+b = 2 \iff b = 0$ $\therefore \bar{v}(t) = (t-1,\, 2\sqrt{t})$

KURVANS LÄNGD ÄR $\int_1^3 |\bar{v}|\,dt = \int_1^3 \sqrt{(t-1)^2 + (2\sqrt{t})^2}\,dt =$

$= \int_1^3 \sqrt{t^2-2t+1+4t}\,dt = \int_1^3 \sqrt{t^2+2t+1}\,dt = \int_1^3 \sqrt{(t+1)^2}\,dt$

$= \int_1^3 |t+1|\,dt = \int_1^3 (t+1)\,dt = \left[\frac{t^2}{2}+t\right]_1^3 = \frac{9}{2}+3-\frac{1}{2}-1 = 6$

<u>SVAR</u>: 6 l.e

8. $z + xe^z = x^2+y^2$ $\quad z(1,0) = 0$

$\frac{\partial z}{\partial y}$: $z'_y + x\cdot z'_y e^z = 2y$ $\quad z'_y(1+xe^z) = 2y$ $\quad z'_y = \frac{2y}{1+xe^z}$

$z'_y(1,0) = 0$

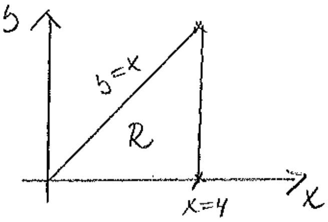
$\frac{\partial z}{\partial x}$: $z'_x + e^z + xz'_x e^z = 2x$ $\quad z'_x(1+xe^z) = 2x - e^z$

$z'_x = \frac{2x-e^z}{1+xe^z}$ $\quad z'_x(1,0) = \frac{2-1}{1+1} = \frac{1}{2}$

$\frac{\partial^2 z}{\partial x\partial y}$: $z''_{xy} + z'_y e^z + xz''_{xy}e^z + xz'_y\cdot z'_x e^z = 0$

$z''_{xy}(1+xe^z) = -z'_y e^z - xz'_y\cdot z'_x e^z$ $\quad \therefore z''_{xy}(1,0) = 0$; <u>SVAR</u>

9. $I = \iint_R \frac{\sqrt{x+y}}{x}\,dx\,dy$



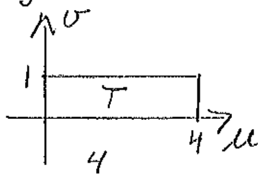
$$\begin{cases} x = u \\ y = u \cdot v \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u \Rightarrow dx\,dy = u\,du\,dv$$

$x = 4$ ÖVERFÖRS TILL $u = 4$

$y = x \Leftrightarrow y - x = 0$ ÖVERFÖRS TILL $uv - u = 0 \Leftrightarrow u(v-1) = 0$
$u = 0,\ v = 1$

$y = 0$ ÖVERFÖRS TILL $u = 0$ OCH $v = 0$



$$I = \iint_T \frac{\sqrt{u + u\cdot v}}{u} \cdot u\,du\,dv = \int_0^1\int_0^4 \sqrt{u(1+v)}\,du\,dv$$

$$= \int_0^4 \sqrt{u}\,du \cdot \int_0^1 \sqrt{1+v}\,dv = \left[\frac{2}{3}u\sqrt{u}\right]_0^4 \cdot \left[\frac{2}{3}(1+v)\sqrt{1+v}\right]_0^1 =$$

$$= \frac{2}{3}(4\cdot 2)\cdot\frac{2}{3}(2\sqrt{2}-1) = \frac{32}{9}(2\sqrt{2}-1)$$

SVAR: $\frac{32}{9}(2\sqrt{2}-1)$

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10. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ A SYMMETRISK $\Rightarrow b = c$

KAR. EKV: $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Leftrightarrow (a-\lambda)(d-\lambda) - bc = 0$

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - \lambda(a+d) + ad - bc = 0$$

$$\lambda = \frac{1}{2}(a+d) \pm \sqrt{\frac{1}{4}(a+d)^2 - ad + bc}$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{a^2+2ad+d^2-4ad+4bc}{4}} = \frac{a+d}{2} \pm \sqrt{\frac{a^2-2ad+d^2+4bc}{4}}$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{(a-d)^2+4bc}{4}} = |b=c| = \frac{a+d}{2} \pm \sqrt{\frac{(a-d)^2+4b^2}{4}}$$

$(a-d)^2 + 4b^2 \geq 0 \quad \therefore \lambda$ HAR ENDAST REELLA VÄRDEN

V.S.V.