1a) $\bar{u}=(1,2,a)$, $\bar{v}=(0,1,2)$, $\bar{w}=(1,4,1)$, $V=|\bar{u}\cdot(\bar{v}\times\bar{w})|=1$

$$\bar{u}\cdot(\bar{v}\times\bar{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & a \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{vmatrix} = \{\text{utv. längs 1:a kol}\}$$

$$= 1\cdot\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & a \\ 1 & 2 \end{vmatrix} = 1-8+4-a = -3-a$$

$|-3-a|=1$ $\quad -3-a=1 \iff a=-4$
$\quad 3+a=1 \iff a=-2$

<u>SVAR</u>! $a=-2$ ELLER $a=-4$

1b) $p=(2,3,1)$ $\quad \pi: x+y-z=7$ $\quad \bar{n}_\pi=(1,1,-1)$

LINJEN GENOM p VINKELRÄT MOT π HAR EKV.

$$l=\begin{cases} x=2+t \\ y=3+t \\ z=1-t \end{cases}$$

l SKÄR π I PUNKTEN q

$\Rightarrow 2+t+3+t-1+t=7$
$3t+4=7$
$t=1$ SÄTTS IN I l

$\therefore q=(3,4,0)$. $\quad$ <u>SVAR</u>: $q=(3,4,0)$

2a) $A=k\begin{pmatrix} 8 & 1 & a \\ 1 & 8 & b \\ 4 & -4 & c \end{pmatrix}$

INGÅENDE RADER/KOLONNVEKTORER SKALL VARA ORTOGONALA OCH HA LÄNGDEN 1.

$\bar{k}_1=(8k,k,4k)$ $\quad |\bar{k}_1|=\sqrt{64k^2+k^2+16k^2}=\sqrt{81k^2}=|9k|$

$k>0 \Rightarrow 9k=1$ $\quad k=\frac{1}{9}$

$\bar{r}_1=k(8,1,a)=\frac{1}{9}(8,1,a)$ $\quad |\bar{r}_1|=\frac{1}{9}\sqrt{64+1+a^2}=\frac{1}{9}\sqrt{65+a^2}$

$\frac{1}{9}\sqrt{65+a^2}=1 \iff 65+a^2=81 \iff a=\pm 4$

P.S.S $\bar{r}_2=k(1,8,b)=1$ $\quad b=\pm 4$

$\bar{r}_3=k(4,-4,c)=\frac{1}{9}(4,-4,c)$ $\quad |\bar{r}_3|=\frac{1}{9}\sqrt{16+16+c^2}=\frac{1}{9}\sqrt{32+c^2}$

$32+c^2=81 \iff c=\pm 7$ $\quad c>0 \Rightarrow c=7$.

$\bar{r}_1\cdot\bar{r}_3=0 \iff (8,1,a)(4,-4,7)=32-4+7a=0 \iff a=-4$

FORTS

2a forts.

$$\bar{r}_2 \cdot \bar{r}_3 = 0 \iff (1, 8, b)\cdot(4,-4,7) = 4-32+7b = 0$$

$\iff b = 4.$ SVAR: $a=-4$, $b=4$, $c=7$, $k=\frac{1}{9}$

2b)

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = \begin{vmatrix} c & -c & a-b \\ b & a+c & b \\ c & c & a+b \end{vmatrix} = \begin{vmatrix} 2c & 0 & 2a \\ b & a+c & b \\ c & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 2c & 0 & 2a \\ b-c & a & -a \\ c & c & a+b \end{vmatrix} = \{\text{UTV. LÄNGS 2:A KOL}\} =$$

$$= a\cdot(-1)^{2+2}\cdot\begin{vmatrix} 2c & 2a \\ c & a+b \end{vmatrix} + c\cdot(-1)^{3+2}\begin{vmatrix} 2c & 2a \\ b-c & -a \end{vmatrix} =$$

$$= a(2ac+2bc-2ac) - c(-2ac-2ab+2ac) = 4abc \quad \text{VSV.}$$

3a)

$$\sum_{n=0}^{\infty} \frac{\sqrt{n}}{\sqrt{n^3+1}} \qquad a_n = \frac{\sqrt{n}}{\sqrt{n^3+1}} \qquad \text{LÅT } b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n^3+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{n\sqrt{n}\left(\sqrt{1+\frac{1}{n^3}}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^3}}} = 1 \qquad \sum \frac{1}{n} \text{ ÄR DIV. TY } \sum \frac{1}{n^a} \text{ DIV}$$

DÅ $a=1$ $\Rightarrow \sum \frac{\sqrt{n}}{\sqrt{n^3+1}}$ DIVERGENT ENL.

JÄMFÖRELSEPRINCIPEN. SVAR: DIVERGENT

3b)

$$z(x,y) = u\left(\frac{x}{y}\right) \qquad \frac{\partial z}{\partial x} = \frac{1}{y}u'\left(\frac{x}{y}\right) \qquad \frac{\partial^2 z}{\partial x^2} = \frac{1}{y^2}u''\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = -\frac{x}{y^2}u'\left(\frac{x}{y}\right) \qquad \frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3}u'\left(\frac{x}{y}\right) + \frac{x^2}{y^4}u''\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{y^2}u'\left(\frac{x}{y}\right) - \frac{x}{y^3}u''\left(\frac{x}{y}\right)$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{y^2}u'' + 2xy\left(-\frac{1}{y^2}u' - \frac{x}{y^3}u''\right) + y^2\left(\frac{2x}{y^3}u' + \frac{x^2}{y^4}u''\right)$$

$$= \left(\frac{x^2}{y^2} - \frac{2x^2}{y^2} + \frac{x^2}{y^2}\right)u'' + \left(-\frac{2x}{y} + \frac{2x}{y}\right)u' = 0 \quad \text{VSV.}$$

4a) $f(x,y,z) = x^3 + yz^2 + y^2 \qquad P_0 = (1,1,2)$

$F: \; z = \sqrt{4xy} = 2\sqrt{xy} \iff z - 2\sqrt{xy} = 0$

$\text{GRAD } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (3x^2, z^2+2y, 2yz)$

$\text{GRAD } f(1,1,2) = (3, 4+2, 2\cdot 1\cdot 2) = (3, 6, 4)$

$\text{GRAD } F = \left(-\frac{2\cdot\sqrt{y}}{2\sqrt{x}}, -\frac{2\sqrt{x}}{2\sqrt{y}}, 1\right) \qquad \text{GRAD } F(1,1,2) = (-1,-1,1)$

$\bar{e} = (-1,-1,1) \qquad |\bar{e}| = \sqrt{3}$

$$\frac{df}{de} = \text{GRAD } f \cdot \frac{\bar{e}}{|\bar{e}|} = (3,6,4)\cdot\frac{1}{\sqrt{3}}(-1,-1,1) = \frac{1}{\sqrt{3}}(-3-6+4) = \frac{-5}{\sqrt{3}}$$

SVAR: $\frac{-5}{\sqrt{3}}$

4b) $f(x,y) = e^{x+y} \sin xy$

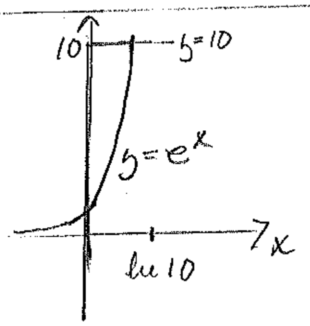
M.L för $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + O(x^4)$

$\sin x = x - \frac{x^3}{3!} + O(x^5)$

$$f(x,y) = \left(1 + (x+y) + \frac{(x+y)^2}{2} + R_2\right)\left(xy - \frac{(xy)^3}{3!} + R_3\right) =$$

$= xy + x^2y + xy^2 + R_3.$ SVAR: $f(x,y) = xy + x^2y + xy^2 + R_3$

5a) $\displaystyle\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx$



$y = e^x \iff x = \ln y$

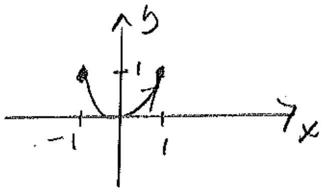
BYT INT. ORDNING!

$$\int_1^{10}\int_0^{\ln y} \frac{1}{\ln y}\, dx\, dy =$$

$$\int_1^{10}\left[\frac{x}{\ln y}\right]_0^{\ln y} dy = \int_1^{10}\left[\frac{\ln y}{\ln y} - 0\right] dy = \int_1^{10} dy = \left[y\right]_1^{10} = 9$$

SVAR: 9.

5b) $\int_{\Gamma} (2xy + y^2)\,dx + (x^3 + y)\,dy =$



$y = x^2$

$x = t \qquad dx = dt \qquad -1 \le t \le 1$
$y = t^2 \qquad dy = 2t\,dt$

$$\int_{-1}^{1} (2t \cdot t^2 + t^4) \cdot dt \; + \; (t^3 + t^2) \cdot 2t\,dt =$$

$$\int_{-1}^{1} (2t^3 + t^4 + 2t^4 + 2t^3)\,dt = \int_{-1}^{1} (3t^4 + 4t^3)\,dt =$$

$$\left[\frac{3t^5}{5} + t^4\right]_{-1}^{1} = \frac{3}{5} + 1 - \left(-\frac{3}{5} + 1\right) = \frac{6}{5}.$$

<u>SVAR:</u> $\frac{6}{5}$

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6. $z = \arctan \frac{x}{y} \qquad \Pi: 2x + y - 5z = 0$

$F: \arctan \frac{x}{y} - z = 0$

$$\bar{n}_F = \text{grad } F = \left(\frac{1}{y} \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2},\ -\frac{x}{y^2} \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2},\ -1\right)$$

$$= \left(\frac{1}{y} \cdot \frac{y^2}{y^2+x^2},\ -\frac{x}{y^2} \cdot \frac{y^2}{y^2+x^2},\ -1\right) = \left(\frac{y}{x^2+y^2},\ \frac{-x}{x^2+y^2},\ -1\right)$$

$\bar{n}_{\Pi} = (2, 1, -5) \qquad \bar{n}_F \parallel \bar{n}_{\Pi} \iff \bar{n}_F = k \cdot \bar{n}_{\Pi}$

$$\begin{cases} \text{I} & \frac{y}{x^2+y^2} = k \cdot 2 \\ \text{II} & \frac{-x}{x^2+y^2} = k \\ \text{III} & -1 = -5k \end{cases}$$

III GER $k = \frac{1}{5}$

$\frac{\text{I}}{\text{II}}: \frac{y}{-x} = 2 \iff y = -2x$

$y = -2x$ i F: $\arctan \frac{x}{-2x} - z = 0$

$$z = \arctan\left(-\tfrac{1}{2}\right) = -\arctan \frac{1}{2}$$

$y = -2x$ i I $\quad \frac{-2x}{x^2 + 4x^2} = 2 \cdot \frac{1}{5} \iff -\frac{2}{5x} = \frac{2}{5} \iff x = -1$

$x = -1 \Rightarrow y = 2.$ PUNKTEN ÄR $(-1, 2, -\arctan \frac{1}{2})$

<u>SVAR:</u> $(-1, 2, -\arctan \frac{1}{2})$

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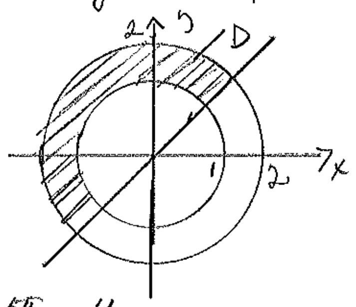
7. $\iint_D \frac{x^2}{1+(x^2+y^2)^2}\,dx\,dy \qquad 1 \le x^2+y^2 \le 4 \;,\; x \le y$



$$\left| \begin{array}{ll} \text{POLÄRA KOORDINATER!} & \\ x = r\cos\theta & 1 \le r \le 2 \\ y = r\sin\theta & \frac{\pi}{4} \le \theta \le \frac{5\pi}{4} \\ dx\,dy = r\,dr\,d\theta & \end{array} \right| =$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_1^4 \frac{r^2\cos^2\theta \cdot r\,dr\,d\theta}{1+(r^2)^2} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_1^4 \frac{r^3\cos^2\theta}{1+r^4}\,dr\,d\theta =$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos^2\theta\,d\theta \cdot \int_1^4 \frac{r^3}{1+r^4}\,dr = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1+\cos 2\theta}{2}\,d\theta \cdot \left[\frac{\ln|1+r^4|}{4}\right]_1^4 =$$

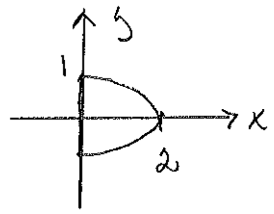
$$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cdot \left(\frac{\ln 17 - \ln 2}{4}\right) = \left(\frac{5\pi}{8} + \frac{1}{4} - \frac{\pi}{8} - \frac{1}{4}\right)\cdot \frac{1}{4}\ln\frac{17}{2}$$

$$= \frac{\pi}{2}\cdot\frac{1}{4}\ln\frac{17}{2} = \frac{\pi}{8}\ln\frac{17}{2}. \qquad \underline{\text{SVAR!}}\; \frac{\pi}{8}\ln\frac{17}{2}$$

---

8. $f(x,y) = x^2+y^2-x+1 \qquad x^2+4y^2 \le 4 \iff \frac{x^2}{2^2}+y^2 \le 1$

$x \ge 0$



INRE STAT. PUNKTER!

$f'_x = 2x-1 \qquad f'_x = 0 \Rightarrow x = \frac{1}{2}$

$f'_y = 2y \qquad f'_y = 0 \Rightarrow y = 0$

$(x,y) = (\frac{1}{2}, 0) \qquad f(\frac{1}{2},0) = \frac{1}{4} - \frac{1}{2} + 1 = \underline{\frac{3}{4}}.$ MINST

RANDEN: 1) $x = 0 \Rightarrow f(0,y) = y^2+1 = h(y)$

$h'(y) = 2y \qquad h' = 0 \Rightarrow y = 0 \qquad h(0) = \underline{1}$

2) $y^2 = 1 - \frac{x^2}{4} \qquad f\left(x, \pm\sqrt{1-\frac{x^2}{4}}\right) = x^2 + 1 - \frac{x^2}{4} - x + 1 = \frac{3x^2}{4} - x + 2 = g(x)$

$g'(x) = \frac{3x}{2} - 1 \qquad g' = 0 \Rightarrow x = \frac{2}{3} \qquad g(\frac{2}{3}) = \frac{3}{4}\cdot\frac{4}{9} - \frac{2}{3} + 2 = \underline{\underline{\frac{5}{3}}}$

3) HÖRN $(0,1), (0,-1), (2,0) \qquad f(0,1) = \underline{\underline{2}} \qquad f(0,-1) = \underline{\underline{2}}$

$f(2,0) = 4 - 2 + 1 = \underline{3}$ STÖRST

<u>SVAR!</u> STÖRSTA VÄRDET ÄR 3
MINSTA VÄRDET ÄR $\frac{3}{4}$

9.

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

A ÄR SYMMETRISK MATRIS

SÖK EGENVÄRDEN TILL A

KAR EKV:

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -\lambda & -2 \\ -2 & -\lambda & -\lambda \\ -1 & -1 & 1-\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} -\lambda & 0 & -2 \\ -2 & -\lambda+2 & -\lambda \\ -1 & 0 & 1-\lambda \end{vmatrix} = \{\text{UTV. LÄNGS 2:a KOL}\} =$$

$$= (-\lambda+2)(-1)^4 \begin{vmatrix} -\lambda & -2 \\ -1 & 1-\lambda \end{vmatrix} = (2-\lambda)(-\lambda(1-\lambda)-2) = (2-\lambda)(\lambda^2-\lambda-2)$$

$$= (2-\lambda)(\lambda-2)(\lambda+1) = 0 \iff \lambda_1 = -1,\ \lambda_{2,3} = 2 \text{ (DUBBELROT)}$$

SÖK EGENVEKTORER:

$\lambda = -1$

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right) \quad \begin{matrix} 3y = 3z \\ y = z \end{matrix} \quad x = 2y - z = z \quad \bar{v}_1 = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \end{array}\right) \quad \left.\begin{matrix} x = -y-z \\ y = t \\ z = s \end{matrix}\right\} \Rightarrow$$

$$\bar{v} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \bar{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

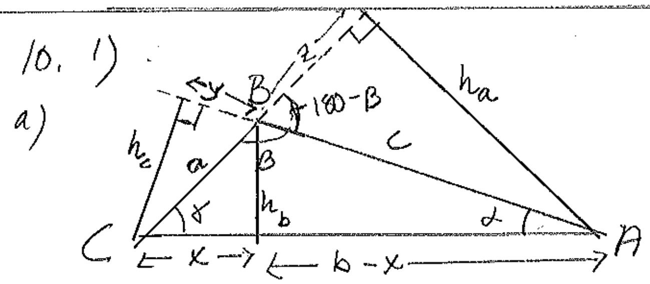
$\bar{v}_1 \cdot \bar{v}_2 = 0 \quad \bar{v}_1 \cdot \bar{v}_3 = 0 \quad \bar{v}_2 \cdot \bar{v}_3 \neq 0$ SÖK EN VEKTOR SOM ÄR ORTOGONAL MOT $\bar{v}_1$ OCH $\bar{v}_2$

$$\bar{v}_4 = \bar{v}_1 \times \bar{v}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (0-1, -1, 1+1) = (-1, -1, 2) \quad \therefore \quad \begin{matrix} \bar{v}_4 \cdot \bar{v}_1 = 0 \\ \bar{v}_4 \cdot \bar{v}_2 = 0 \end{matrix}$$

$$|\bar{v}_1| = \sqrt{3} \quad |\bar{v}_2| = \sqrt{2} \quad |\bar{v}_4| = \sqrt{6} \qquad P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} = \underline{\text{SVAR}}$$

10. 1)

a)



DELA UPP SIDAN b M.H.A. HÖJDEN h_b

$$\left.\begin{aligned} \cos\gamma &= \frac{x}{a} \Leftrightarrow x = a\cos\gamma \\ \cos\alpha &= \frac{b-x}{c} \Leftrightarrow b-x = c\cos\alpha \end{aligned}\right\} \text{ADDERA}$$

TILLS. $b = a\cos\gamma + c\cos\alpha$

2) RITA HÖJDEN h_c MOT SIDAN c !

$$\left.\begin{aligned} \cos\alpha &= \frac{y+c}{b} \Leftrightarrow b\cos\alpha = y+c \\ \underbrace{\cos(180-\beta)}_{-\cos\beta} &= \frac{y}{a} \Leftrightarrow -a\cos\beta = y \end{aligned}\right\} \Rightarrow$$

SUBTRAHERA:

$$c = b\cos\alpha - (-a\cos\beta)$$
$$c = b\cos\alpha + a\cos\beta$$

3) RITA HÖJDEN h_a MOT SIDAN a !

$$\left.\begin{aligned} \cos\gamma &= \frac{a+z}{b} \Leftrightarrow b\cos\gamma = a+z \\ \cos(180-\beta) &= \frac{z}{c} \Leftrightarrow -c\cos\beta = z \end{aligned}\right\}$$

SUBTRAHERA:

$$a = b\cos\gamma + c\cos\beta$$

$$(I): \begin{cases} c\cos\alpha + a\cos\gamma = b \\ b\cos\alpha + a\cos\beta = c \\ c\cos\beta + b\cos\gamma = a \end{cases}$$

V.S.V.

b) KALLA $\cos\alpha = R$, $\cos\beta = S$, $\cos\gamma = T$. DÅ BLIR

$$(I): \begin{cases} cR + aT = b \\ bR + aS = c \\ cS + bT = a \end{cases}$$

LÖS UT T.

$$\underbrace{\begin{pmatrix} c & 0 & a \\ b & a & 0 \\ 0 & c & b \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} R \\ T \\ S \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} b \\ c \\ a \end{pmatrix}}_{\bar{b}}$$

$\det A = abc + abc = 2abc$

$$T = \frac{\begin{vmatrix} c & 0 & b \\ b & a & c \\ 0 & c & a \end{vmatrix}}{\det A} =$$

$$= \frac{a^2c + b^2c - c^3}{2abc} = \frac{c(a^2+b^2-c^2)}{2abc} = \frac{a^2+b^2-c^2}{2ab}$$

$$\therefore \cos\gamma = \frac{a^2+b^2-c^2}{2ab} \Leftrightarrow 2ab\cos\gamma = a^2+b^2-c^2$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

(COSINUS-SATSEN)