FÖRSLAG TILL LÖSNING TENT. I MATEMATIK II FÖR CL.
SF1613. 2008-08-26.

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1. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{pmatrix}$ INVERS EXISTERAR DÅ $\det A \neq 0$

$$\det A = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{vmatrix} = \{\text{UTV. LÄNGS 2:A RAD}\} = 1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 1 & 0 \\ 2 & a \end{vmatrix} = -a$$

$\det A \neq 0$ DÅ $a \neq 0$.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & a & 0 & 0 & 1 \end{array}\right) \updownarrow \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & a & 0 & 0 & 1 \end{array}\right) \overset{(-1)}{\swarrow} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 2 & a & 0 & -1 & 1 \end{array}\right) \overset{(-2)}{\downarrow}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & a & -2 & 1 & 1 \end{array}\right) \cdot \frac{1}{a} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{array}\right)$$

$\underbrace{\qquad\qquad}_{A^{-1}}$

SVAR: $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{pmatrix}$, $a \neq 0$

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2. $x^2 - 2xy + 3y^2 = 1 \quad (*)$

SKRIV OM $x^2 - 2xy + 3y^2$ PÅ FORMEN $\frac{x^2}{a^2} + \frac{y^2}{b^2}$

DVS HUVUDAXELFORM. BESTÄM EGENVÄRDEN TILL

$A = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$ KAR. EKV: $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 1 = 0$

$3 - 4\lambda + \lambda^2 - 1 = 0 \iff \lambda = 2 \pm \sqrt{2}$

$(*)$ KAN SKRIVAS SOM $(2+\sqrt{2})u^2 + (2-\sqrt{2})v^2 = 1$

$$\frac{u^2}{\left(\frac{1}{\sqrt{2+\sqrt{2}}}\right)^2} + \frac{v^2}{\left(\frac{1}{\sqrt{2-\sqrt{2}}}\right)^2} = 1 \quad \therefore a = \frac{1}{\sqrt{2+\sqrt{2}}} \quad b = \frac{1}{\sqrt{2-\sqrt{2}}}$$

SÖKTA AREAN ÄR: $\pi ab = \pi \cdot \frac{1}{\sqrt{2+\sqrt{2}}} \cdot \frac{1}{\sqrt{2-\sqrt{2}}} =$

$$= \frac{\pi}{\sqrt{(2+\sqrt{2})(2-\sqrt{2})}} = \frac{\pi}{\sqrt{4-2}} = \frac{\pi}{\sqrt{2}}$$

SVAR: $\frac{\pi}{\sqrt{2}}$

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3. $$\sum_{n=0}^{\infty} \frac{n^2 x^n}{n^2+1} = \sum_{n=0}^{\infty} a_n x^n \qquad a_n = \frac{n^2}{n^2+1}$$

SÖK KONVERGENSRADIEN R

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{n^2}{n^2+1}}{\frac{(n+1)^2}{(n+1)^2+1}} = \lim_{n\to\infty}\frac{n^2\cdot((n+1)^2+1)}{(n^2+1)(n+1)^2} =$$

$$= \lim_{n\to\infty} \frac{n^2(n^2+2n+2)}{(n^2+1)(n^2+2n+1)} = \lim_{n\to\infty} \frac{n^4(1+\frac{2}{n}+\frac{2}{n^4})}{n^4(1+\frac{2}{n}+\frac{2}{n^2}+\frac{2}{n^3}+\frac{1}{n^4})} = 1$$

$\therefore$ SERIEN KONV. DÅ $|x| < 1$. UNDERSÖK $x=1$ <u>O</u> $x=-1$

$x=1 \Rightarrow \sum_{n=0}^{\infty} \frac{n^2}{n^2+1} \qquad \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0 \quad \therefore$ DIV.

$x=-1 \Rightarrow \sum \frac{(-1)^n n^2}{n^2+1}$

1) ALT.

2) $\lim_{n\to\infty} |a_n| \neq 0 \quad \therefore$ DIV ENL. LEIBNIZ.

<u>SVAR</u>: $\sum_{n=0}^{\infty} \frac{n^2 x^n}{n^2+1}$ KONV. DÅ $-1 < x < 1$

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4. $$z = x f(2x+y) + y\, g(x+ay)$$

$$z'_x = f(2x+y) + 2x f'(2x+y) + y g'(x+ay)$$

$$z''_{xx} = 2f'(2x+y) + 2f'(2x+y) + 4x f''(2x+y) + y g''(x+ay)$$
$$= 4f' + 4x f'' + y g''$$

$$z''_{xy} = f'(2x+y) + 2x f''(2x+y) + g'(x+ay) + ay\, g''(x+ay)$$
$$= f' + 2x f'' + g' + ay\, g''$$

$$z'_y = x f'(2x+y) + g(x+ay) + ay\, g'(x+ay)$$

$$z''_{yy} = x f''(2x+y) + a g'(x+ay) + a g'(x+ay) + a^2 y g''(x+ay)$$
$$= x f'' + 2a g' + a^2 y g''$$

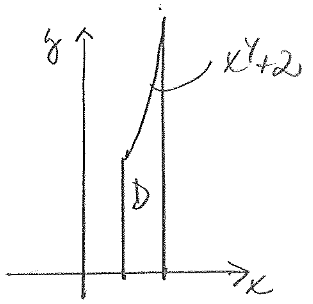
$z''_{xx} - 4z''_{xy} + z''_{yy} = 0 \quad x=y=0$ GER

$$4f' - 4(f'+g') + 2a g' = 0 \iff -4g' + 2ag' = 0$$

$$\iff (2a-4) g' = 0 \quad \text{DVS} \quad 2a-4=0 \iff a=2$$

<u>SVAR</u>: $a=2$

5. $\iint_D \frac{dxdy}{\sqrt{x^4y+1}}$ $\quad 0 \le y \le x^4+2 \ , \ 1 \le x \le 2$



$$\iint_D \frac{dxdy}{\sqrt{x^4y+1}} = \int_1^2 \int_0^{x^4+2} \frac{dy}{\sqrt{x^4y+1}}\, dx =$$

$$= \int_1^2 \left[\frac{2\sqrt{x^4y+1}}{x^4}\right]_0^{x^4+2} dx = 2\int_1^2 \left(\frac{\sqrt{x^4(x^4+2)+1}}{x^4} - \frac{1}{x^4}\right) dx =$$

$$= 2\int_1^2 \left(\frac{\sqrt{x^8+2x^4+1}}{x^4} - \frac{1}{x^4}\right) dx = 2\int_1^2 \left(\frac{\sqrt{(x^4+1)^2}}{x^4} - \frac{1}{x^4}\right) dk =$$

$$= 2\int_1^2 \left(\frac{x^4+1-1}{x^4}\right) dx = 2\int_1^2 1\, dx = 2\left[x\right]_1^2 =$$

$$= 2(2-1) = 2$$

SVAR: 2

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6. $y = A + Bx + Cx^2 + Dx^3$ $\quad (-1,4)\ ,\ (1,2)\ ,\ (2,1)\ ,\ (3,16)$

$$\left.\begin{array}{lll} (-1,4) & \text{GER} & 4 = A - B + C - D \\ (1,2) & \text{GER} & 2 = A + B + C + D \\ (2,1) & \text{GER} & 1 = A + 2B + 4C + 8D \\ (3,16) & \text{GER} & 16 = A + 3B + 9C + 27D \end{array}\right\} \text{GAUSSEL.} \Rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 1 \\ 1 & 3 & 9 & 27 & 16 \end{array}\right)\begin{matrix}(-1)\\ \\ \\ \\ \end{matrix} \sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 2 & 0 & 2 & -2 \\ 0 & 3 & 3 & 9 & -3 \\ 0 & 4 & 8 & 28 & 12 \end{array}\right)\begin{matrix} \\ \cdot\frac{1}{2} \\ \cdot\frac{1}{3} \\ \cdot\frac{1}{4}\end{matrix} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 1 & 2 & 7 & 3 \end{array}\right)\begin{matrix} \\ (1)\ (-1) \\ \\ \\ \end{matrix} \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 & 4 \end{array}\right)\begin{matrix} \\ \\ (-2) \\ \\ \end{matrix}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right)\begin{matrix} \\ \\ (-1) \\ \cdot\frac{1}{2} \end{matrix} \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)\begin{matrix} \\ \\ \\ (-2)\ (-1)\ (2) \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$A = 7$
$B = -3$
$C = -4$
$D = 2$

SVAR:
$y = 7 - 3x - 4x^2 + 2x^3$

7. $F = \ln(x^2+y^2+z^2) = \ln(r^2)$ $\quad P_0 = (1,1,1)$

$E: x^2+y^2+2z^2 = 4$

$$\frac{dF}{d\bar{e}} = \text{grad}\,F \cdot \frac{\bar{e}}{|\bar{e}|}$$

$$\text{grad}\,F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(\frac{2x}{r^2}, \frac{2y}{r^2}, \frac{2z}{r^2}\right) = \frac{2}{r^2}(x,y,z)$$

$$\text{grad}\,F(1,1,1) = \frac{2}{3}(1,1,1)$$

$$\bar{e} = \bar{n}_E(1,1,1) \cdot \quad \bar{n}_E = \text{grad}\,E = (2x, 2y, 4z)$$

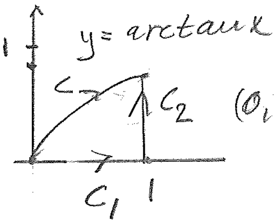
$$\bar{n}_E(1,1,1) = (2,2,4) = 2(1,1,2) \qquad |\bar{n}_E| = 2\sqrt{6} = |\bar{e}|$$

$$\frac{\bar{e}}{|\bar{e}|} = \frac{2(1,1,2)}{2\sqrt{6}} = \frac{(1,1,2)}{\sqrt{6}} \qquad \frac{dF}{d\bar{e}} = \frac{2}{3}(1,1,1)\cdot\frac{(1,1,2)}{\sqrt{6}} = \frac{2}{3\sqrt{6}}(1+1+2)$$

$$= \frac{2}{3\sqrt{6}} \cdot 4 = \frac{8}{3\sqrt{6}} = \frac{4\sqrt{6}}{9}$$

SVAR: $\frac{4\sqrt{6}}{9}$

8. $$\int_C y^2x\,dx + (\cos y + yx^2)\,dy$$



$(0,0) \to (1, \frac{\pi}{4})$

$P = y^2x \qquad Q = \cos y + yx^2$

$$\frac{\partial P}{\partial y} = 2yx \qquad \frac{\partial Q}{\partial x} = 2yx \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$ ∴ VI FÅR BYTA VÄG.

VÄLJ NY VÄG t.ex. $C_1 + C_2$

$C_1: \begin{cases} x = t & dx = dt \\ y = 0 & dy = 0 \\ 0 \le t \le 1 \end{cases}$ $\qquad C_2: \begin{matrix} x = 1 & dx = 0 \\ y = t & dy = dt \\ 0 \le t \le \frac{\pi}{4} \end{matrix}$

$$\int_C = \int_{C_1} + \int_{C_2} = \int_0^1 0 + \int_1^{\frac{\pi}{4}} (0 + \cos t + t)\,dt$$

$$= \left[\sin t + \frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \sin\frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{1}{\sqrt{2}} + \frac{\pi^2}{32}$$

SVAR: $\frac{1}{\sqrt{2}} + \frac{\pi^2}{32}$

9.

$$z = x^3 - y^3 + bxy \quad , \quad b \neq 0$$

$$z'_x = 3x^2 + by \qquad z'_x = 0 \Rightarrow 3x^2 + by = 0 \Leftrightarrow y = -\frac{3x^2}{b} \quad \left\{ \begin{matrix} I \\ II \end{matrix} \right.$$

$$z'_y = -3y^2 + bx \qquad z'_y = 0 \Rightarrow bx - 3y^2 = 0$$

I INSATT I II GER $bx - 3\left(\frac{-3x^2}{b}\right)^2 = 0 \Leftrightarrow$

$$bx - 3 \cdot \frac{9x^4}{b^2} = 0 \Leftrightarrow b^3x - 27x^4 = 0 \Leftrightarrow x(b^3 - 27x^3) = 0$$

1) $x = 0$, 2) $27x^3 = b^3$

$x = \frac{b}{3}$.

$x = 0 \Rightarrow y = 0$ OCH $x = \frac{b}{3} \Rightarrow y = \frac{-3 \cdot \frac{b^2}{9}}{b} = -\frac{b}{3}$

$\therefore$ STAT. PUNKTER ÄR $(0,0)$ OCH $\left(\frac{b}{3}, -\frac{b}{3}\right)$

$z''_{xx} = 6x$

$z''_{xy} = b$

$z''_{yy} = -6y$

PKT	A	B	C	$AC-B^2$
$(0,0)$	0	b	0	$-b^2$
$\left(\frac{b}{3}, \frac{-b}{3}\right)$	$2b$	b	$2b$	$4b^2 - b^2 = 3b^2$

$-b^2 < 0$ $\therefore$ $(0,0)$ ÄR EN SADELPUNKT

$3b^2 > 0$ $\therefore$ $\left(\frac{b}{3}, -\frac{b}{3}\right)$ ÄR ETT LOK. MIN DÅ $b > 0$

OCH $\left(\frac{b}{3}, -\frac{b}{3}\right)$ " " LOK. MAX DÅ $b < 0$.

<u>SVAR</u>: $z(0,0) = 0$ SADELPUNKT

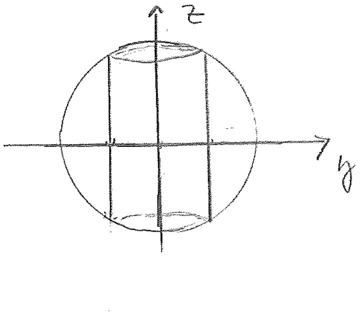
$z\left(\frac{b}{3}, -\frac{b}{3}\right) = -\frac{b^3}{27}$ LOK. MINPUNKT DÅ $b > 0$

" MAXPUNKT DÅ $b < 0$

10.

$$\begin{cases} x^2 + y^2 + z^2 \leq 4 \\ x^2 + y^2 = 1 \end{cases}$$

BESTÄM FÖRST VOLYMEN INNANFÖR CYLINDERN $= V_I$



V_I : $-\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}$

$D = \{(x,y) : x^2 + y^2 \leq 1\}$

$$V_I = \iint_D \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dx dy =$$

$$= \iint_D 2\sqrt{4-x^2-y^2}\, dx dy = \left| \begin{matrix} \text{POL. KOORD} \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{matrix} \right| = 2\int_0^{2\pi}\int_0^1 \sqrt{4-r^2}\, r dr d\theta$$

FORTS

10 FORTS. IFRÅN 5C)

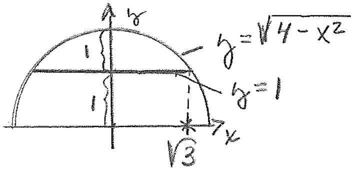
$$= 2\int_0^{2\pi} d\theta \int_0^1 \sqrt{4-r^2}\cdot r\,dr = \left| \begin{array}{l} \sqrt{4-r^2} = t \\ 4-r^2 = t^2 \\ -2r\,dr = 2t\,dt \\ r=1 \Rightarrow t=\sqrt{3} \\ r=0 \Rightarrow t=2 \end{array} \right| = 2\cdot 2\pi \cdot \int_2^{\sqrt{3}} -t^2\,dt = -4\pi\cdot\left[\frac{t^3}{3}\right]_2^{\sqrt{3}}$$

$$= -4\pi\left(\frac{3\sqrt{3}}{3} - \frac{8}{3}\right) = 4\pi\left(\frac{8}{3} - \sqrt{3}\right)$$

SÖKTA VOLYMEN: $V_{KLOT} - V_I = \frac{4\pi\cdot 2^3}{3} - 4\pi\left(\frac{8}{3} - \sqrt{3}\right) = 4\pi\sqrt{3}$

(ALT. SÄTT ATT BERÄKNA V: ANVÄND ROTATION AV HALVCIRKEL O LINJE

ROT. KRING X-AK.



$$V_I = 2\left(\pi\int_0^{\sqrt{3}} (\sqrt{4-x^2})^2\,dx - \pi\int_0^{\sqrt{3}} 1^2\,dx\right) = 4\pi\sqrt{3}$$)

SVAR: $4\pi\sqrt{3}$ v.e

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11. ANTAG ATT MATRISEN A ÄR EN $n \times n$ MATRIS.

KAR. EKV BERÄKNAS GENOM

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & & & a_{nn}-\lambda \end{vmatrix} =$$

$$= \lambda^n + c_1\lambda^{n-1} + \dots + c_n = 0$$

$\lambda = 0$ ÄR EN LÖSNING OMM $c_n = 0$.

DET RÄCKER ATT VISA ATT A ÄR INV. BAR OMM $c_n \neq 0$,

MEN $\det(A - \lambda I) = \lambda^n + c_1\lambda^{n-1} + \dots + c_n$

OM VI SÄTTER $\lambda = 0$ GER DET

$\det A = c_n$. DÄRAV FÖLJER ATT $\det A = 0$

OMM $c_n = 0$ SOM I SIN TUR MEDFÖR ATT

A ÄR INV. BAR OMM $c_n \neq 0$.

V.S.V.