LÖSNINGSFÖRSLAG TILL TENTAMEN MATEMATIK I FÖR CL

5B1143, TEN2, 2005-12-20.

1.a) $\lim\limits_{x\to\infty}(x\sqrt{x^4+x}-x^3) = \lim\limits_{x\to\infty}\left[x(\sqrt{x^4+x}-x^2)\right] =$

$= \lim\limits_{x\to\infty}\left[x\left(\frac{(\sqrt{x^4+x}-x^2)(\sqrt{x^4+x}+x^2)}{\sqrt{x^4+x}+x^2}\right)\right] =$

$= \lim\limits_{x\to\infty}\frac{x(x^4+x-x^4)}{\sqrt{x^4(1+\frac{1}{x^3})}+x^2} = \lim\limits_{x\to\infty}\frac{x\cdot x}{x^2\sqrt{1+\frac{1}{x^3}}+x^2} =$

$= \lim\limits_{x\to\infty}\frac{x^2}{x^2(\sqrt{1+\frac{1}{x^3}}+1)} = \lim\limits_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^3}}+1} = \frac{1}{1+1} = \frac{1}{2}$

<u>SVAR</u>: $\frac{1}{2}$

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b) $f(x)$ ÄR KONT OM $\lim\limits_{x\to 2} f(x) = f(2)$

$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\frac{x^2-x-2}{x-2} = \lim\limits_{x\to 2}\frac{(x-2)(x+1)}{x-2} =$

$= \lim\limits_{x\to 2} x+1 = 2+1 = 3.$

OM $A=3$ BLIR $f(x)$ KONT. <u>SVAR</u>: $A=3$

2a) $f(x) = x\sqrt{4-x^2}$ $\quad D_f: 4-x^2 \geq 0 \iff x^2 \leq 4 \iff -2\leq x\leq 2$

$f'(x) = \sqrt{4-x^2} + x\cdot\frac{1}{2}\frac{1}{\sqrt{4-x^2}}\cdot -2x =$

$= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{(4-x^2)-x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$

$f'(x) = 0 \Rightarrow 4-2x^2 = 0 \iff x^2 = 2$
$x = \pm\sqrt{2}$

x	-2		$-\sqrt{2}$		$\sqrt{2}$		2
f'		$-$	0	$+$	0	$-$	
f		$\searrow$	LOK MIN	$\nearrow$	LOK MAX	$\searrow$	

$f(-\sqrt{2}) = -\sqrt{2}\sqrt{4-2} = -2$
$f(\sqrt{2}) = \sqrt{2}\sqrt{4-2} = 2$
$f(-2) = 0$, $f(2) = 0$

<u>SVAR</u>: $D_f: -2\leq x\leq 2$, $V_f: -2\leq y\leq 2$ LOK MAX: $f(\sqrt{2})=2$
LOK MIN: $f(-\sqrt{2})=-2$

2b)

$$\int_0^5 \frac{x}{\sqrt{x+4}}\,dx = \left| \begin{array}{l} \sqrt{x+4} = t \\ x+4 = t^2 \\ dx = 2t\,dt \\ x=0 \Rightarrow t=2 \\ x=5 \Rightarrow t=3 \end{array} \right| = \int_2^3 \frac{(t^2-4)\cdot 2t\,dt}{t} =$$

$$2\int_2^3 (t^2-4)\,dt = 2\left[t^3/3 - 4t\right]_2^3 = 2\left((9-12)-\left(\tfrac{8}{3}-8\right)\right) =$$

$$= 2\left(-3+8-\tfrac{8}{3}\right) = 2\left(\frac{24-9-8}{3}\right) = \frac{2\cdot 7}{3} = \frac{14}{3} \quad \text{SVAR: } \frac{14}{3}$$

3a)

$$\begin{cases} x(t) = e^{-t}\cos t \\ y(t) = e^{-t}\sin t \end{cases} \qquad 0 \le t \le \frac{\pi}{2}$$

$$L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (y')^2}\,dt$$

$$x'(t) = \frac{dx}{dt} = -e^{-t}\cdot\cos t - e^{-t}\sin t = -e^{-t}(\cos t + \sin t)$$

$$y'(t) = \frac{dy}{dt} = -e^{-t}\sin t + e^{-t}\cos t = e^{-t}(\cos t - \sin t)$$

$$(x')^2 = e^{-2t}(\sin t - \cos t)^2 = e^{-2t}(\sin^2 t - 2\sin t\cos t + \cos^2 t)$$

$$(y')^2 = e^{-2t}(\cos t - \sin t)^2 = e^{-2t}(\cos^2 t + 2\sin t\cos t + \sin^2 t)$$

$$(x')^2 + (y')^2 = e^{-2t}(2\sin^2 t + 2\cos^2 t) = 2e^{-2t}\cdot 1$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{2e^{-2t}} = \sqrt{2}\,e^{-t}$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{2}\,e^{-t}\,dt = \sqrt{2}\left[-e^{-t}\right]_0^{\frac{\pi}{2}} = \sqrt{2}\left(-e^{-\frac{\pi}{2}} - (-1)\right) =$$

$$= \sqrt{2}\left(1 - e^{-\frac{\pi}{2}}\right) \qquad \underline{\text{SVAR:}}\ \sqrt{2}\left(1 - e^{-\frac{\pi}{2}}\right) \text{ l.e}$$

3b)

$$y'' + y' - 12y = 0 \qquad y(0) = 1, \quad y'(0) = 3$$

KAR. EKV $k^2 + k - 12 = 0$

$$k = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{48}{4}}$$

$$k = -\frac{1}{2} \pm \frac{7}{2}$$

$k_1 = 3$
$k_2 = -4$

$$y = Ae^{-4x} + Be^{3x}$$

$$y(0) = A + B \Rightarrow A + B = 1$$

$$y'(x) = -4Ae^{4x} + 3Be^{3x}$$

$$y'(0) = -4A + 3B \Rightarrow -4A + 3B = 3$$

$$\begin{cases} A + B = 1 \\ -4A + 3B = 3 \end{cases} \quad \text{GER} \quad B = 1 \text{ OCH } A = 0$$

$$\therefore\ y = e^{3x} = \underline{\text{SVAR}}$$

4. $$\lim_{x\to 0} \frac{x\ln(1-x)+\sin^2 x}{x^3} = \lim_{x\to 0} \frac{x\left(-x-\frac{(-x)^2}{2}+O(x^3)\right)+\left(x-\frac{x^3}{3!}+O(x^5)\right)^2}{x^3}$$

$$= \lim_{x\to 0} \frac{-x^2-\frac{x^3}{2}+O(x^4)+x^2+O(x^4)}{x^3} = \lim_{x\to 0} \frac{x^3\left(-\frac{1}{2}+O(x)\right)}{x^3}$$

$$= \lim_{x\to 0} -\frac{1}{2}+O(x) = -\frac{1}{2}$$

<u>SVAR</u>: $-\frac{1}{2}$

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5. $$2\cos^2 y - 2\sin^2 x = 1$$

$\frac{d}{dx}$: $-2\cdot 2\cos y\cdot \sin y\cdot y' - 2\cdot 2\sin x\cos x = 0$

$-y'\cdot\cos y\cdot\sin y - \sin x\cos x = 0$

$$y' = -\frac{\sin x\cos x}{\sin y\cdot\cos y}$$

DÅ $y=x$ BLIR $y' = -\frac{\sin x\cos x}{\sin x\cos x} = -1$ <u>SVAR</u>: $y' = -1$

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6. $$\int \ln\sqrt{9+x^2}\,dx = \int \frac{1}{2}\ln(9+x^2)\,dx = \left|\begin{matrix}\text{PART.}\\ \text{INT}\end{matrix}\right| =$$

$$= \frac{x}{2}\ln(9+x^2) - \int \frac{x}{2}\cdot\frac{2x}{9+x^2}\,dx =$$

$$= \frac{x}{2}\ln(9+x^2) - \int \frac{x^2}{9+x^2}\,dx = \frac{x}{2}\ln(9+x^2) - \int \frac{9+x^2-9}{9+x^2}\,dx$$

$$= \frac{x}{2}\ln(9+x^2) - \int 1 - \frac{9}{9+x^2}\,dx = \frac{x}{2}\ln(9+x^2) - x + \int\frac{9}{9+x^2}\,dx$$

$$\int\frac{9}{9+x^2}\,dx = \left|\begin{matrix}x=3t\\ dx=3dt\end{matrix}\right| = \int\frac{9\cdot 3\,dt}{9+9t^2} = \frac{9}{9}\int\frac{3}{1+t^2}\,dt =$$

$$= 3\arctan t = 3\arctan\frac{x}{3} + C$$

$$\int\ln\sqrt{9+x^2}\,dx = \frac{x}{2}\ln(9+x^2) - x + 3\arctan\frac{x}{3} + C.$$

<u>SVAR</u>: $\frac{x}{2}\ln(9+x^2) - x + 3\arctan\frac{x}{3} + C$

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7 a)

$$f(x) = \ln x + (1-x)^2 + 1 \quad , \; x > 0$$

$$f'(x) = \frac{1}{x} + 2(1-x)\cdot(-1) = \frac{1}{x} - 2 + 2x = \frac{1-2x+2x^2}{x}$$

$$= \frac{2(x^2 - x + \frac{1}{2})}{x} = \frac{2\left[(x-\frac{1}{2})^2 - \frac{1}{4} + \frac{1}{2}\right]}{x} =$$

$$= \frac{2\left[(x-\frac{1}{2})^2 + \frac{1}{4}\right]}{x} = \frac{2(x-\frac{1}{2})^2 + \frac{1}{2}}{x} > 0 \quad \text{DÅ } x > 0$$

V.S.V.

b) VISA $f(x) = \ln x + (1-x)^2 \le 0 \; , \; 0 < x \le 1$

$$f'(x) = \frac{2(x-\frac{1}{2})^2 + \frac{1}{2}}{x} \quad | \text{ SE 7a) } | \quad \text{ALLTID POSITIV DÅ } x > 0$$

$\Rightarrow f(x)$ VÄXER. DET STÖRSTA VÄRDET FINNS DÅ 1 $f(1) = 0 + 0^2 = 0$.

$\therefore f(x) \le 0$

8.

$$0 \le y \le \frac{1}{x\sqrt{x^2+1}} \quad , \; x \ge 1$$

$$V_x = \pi \int y^2 \, dx = \pi \int_1^\infty \left(\frac{1}{x\sqrt{x^2+1}}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2(x^2+1)} \, dx =$$

$$= \lim_{T\to\infty} \int_1^T \frac{1}{x^2(x^2+1)} \, dx = \Bigg| \; \frac{1}{x^2(x^2+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1} =$$

$$= \frac{A(x^2+1) + Bx(x^2+1) + x^2(Cx+D)}{x^2(x^2+1)} = \frac{x^3(B+C) + x^2(A+D) + Bx + A}{x^2(x^2+1)}$$

x^3: $B + C = 0 \Rightarrow C = -B = 0$

x^2: $A + D = 0$

x: $B = 0$

x^0: $A = 1 \Rightarrow D = -1$

$$\therefore \frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$$

$$= \lim_{T\to\infty} \int_1^T \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx = \lim_{T\to\infty} \left[-\frac{1}{x} - \arctan x\right]_1^T =$$

$$= \lim_{T\to\infty} \left(\frac{1}{T} - \arctan T + 1 + \arctan 1\right) = 0 - \frac{\pi}{2} + 1 + \frac{\pi}{4} =$$

$$= 1 - \frac{\pi}{4}.$$

SVAR! $(1 - \frac{\pi}{4})$ v.e

9 a) 1. $\lim_{x \to 0^-} |x| \to 0$ $\qquad \lim_{x \to 0^+} |x| = 0$

2. $\lim_{x \to 0^-} \arctan \frac{1}{x} = \arctan(-\infty) = -\frac{\pi}{2}$

$\lim_{x \to 0^+} \arctan \frac{1}{x} = \arctan \infty = \frac{\pi}{2}$

OLIKA

3. $\lim_{x \to 0^-} \frac{\sin x}{x} = 0$ $\qquad \lim_{x \to 0^+} \frac{\sin x}{x} = 0$

SVAR! $\arctan \frac{1}{x}$ BLIR OLIKA OCH HAR VÄRDENA $\pm \frac{\pi}{2}$, DE ÖVRIGA BLIR ALLA 0.

b) $f(x) < g(x)$. NEJ.

MOTEXEMPEL.

$$\frac{1}{1+x^2} < \frac{2}{1+x^2}$$

$$\lim_{x \to \infty} \frac{1}{1+x^2} = \lim_{x \to \infty} \frac{2}{1+x^2} = 0$$

SVAR: NEJ