Mathematics, KTH
B.Ek

Exam TEN1 for the course
SF2736, DISCRETE MATHEMATICS
Wednesday March 16, 2016, at 8.00–13.00.

Examiner: Bengt Ek, tel: 790 6951 (not during the exam).

Allowed aids: Pen/pencil and rubber (and supplied paper), no electronic devices.

Minimum scores for the grades A–E are given in the table:

Grade	A	B	C	D	E	Fx
Score	32	28	22	18	15	13

Fx is not a passing grade, but may be improved to an E by passing a supplementary exam.

To give full points, the solutions must be clear and well explained. Theorems from the course may (unless otherwise stated) be used without proof, if it is clearly written what they say.

PART I

The score on this part is the least of 15 and the sum of the points given for problems 1–5 and bonus points from the homework assignments (at most 4p).

1) (3p) For which $n \in \mathbb{Z}_+$, $n \geq 2$, is it true that every invertible $x \in \mathbb{Z}_n$ is its own inverse (i.e. for $x \in \mathbb{Z}_n$, $x^2 = 1 \Leftrightarrow x$ is invertible)?

2) (3p) It is true that $1234^{503} \equiv 4083 \pmod{6767}$ (you don't have to show that).
Find a $k \in \mathbb{Z}_+$ such that $4083^k \equiv 1234 \pmod{6767}$.

3) (3p) Disa has 21 (consecutive) days of vacation ahead. She wants to spend 14 of them on discrete maths and 7 on the applied theory of waves (i.e. in the pool). In how many ways can she plan her activities, if the first and the last day shall both be spent on discrete maths, there shall be no two consecutive days in the pool, she has 14 different (distinguishable) chapters (each for one day) discrete maths, which can be studied in any order, and the pool days are all the same (i.e. not distinguishable)?
(Answers may contain integers, factorials, powers and the four elementary arithmetical operations.)

4) (3p) The group G has an element g such that $g^6 = (g^{-1})^6$ and $g \neq 1$ (the identity element of G).
Which of the values 1, 2, 3, ..., 10 are possible for $|G|$, the order of G?

5) (3p) A plane, connected graph forms regions in the plane (one of them being the unbounded region). 1 of the regions is bounded by 10 edges, 3 of them by 5 edges, 4 of them by 4 edges and 7 of them by 3 edges (and there are no other regions).
Find the number of vertices in the graph.

PART II

6) The permutations $\alpha, \beta \in S_5$ (the group of permutations of $\{1, 2, \ldots, 5\}$) are given by

$$\alpha(1) = 5,\ \alpha(2) = 3,\ \alpha(3) = 2,\ \alpha(4) = 4,\ \alpha(5) = 1,$$
$$\beta(1) = 5,\ \beta(2) = 1,\ \beta(3) = 3,\ \beta(4) = 2,\ \beta(5) = 4.$$

a) (1p) Give α and $\alpha\beta$ in cycle notation.
b) (1p) Is $\alpha^6\beta\alpha^{-11}\beta^6$ an even or an odd permutation?
c) (2p) Give (in cycle notation) a $\pi \in S_5$ such that $\{(\beta\pi)^i\}_{i=1}^{6}$ are all distinct.

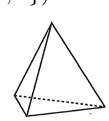
7) (4p) Let $\alpha_1, \alpha_2, \ldots$ be infinite sequences of 0's and 1's (such as $01101000\ldots$). Show that there are $n_i \in \mathbb{Z}_+$, $i = 1, 2, \ldots$ with $n_1 < n_2 < \ldots$, such that all $\alpha_{n_i}, \alpha_{n_{i+1}}, \ldots$ are the same in the first i positions, i.e. all $\alpha_{n_1}, \alpha_{n_2}, \ldots$ start with the same symbol (0 or 1), all $\alpha_{n_2}, \alpha_{n_3}, \ldots$ start with the same two symbols (00,01,10 or 11) etc.

8) (4p) Let $m = \prod_i p_i^{k_i} \neq 1$, where the p_i are distinct primes and the $k_i \in \mathbb{Z}_+$. Further, let $(G, \cdot) = (U_m, \cdot)$ be the group of invertible elements of $\mathbb{Z}_m$.
How many orbits are there when G acts on $\mathbb{Z}_m$ by multiplication?
(Answers may contain the p_i, the k_i, integers, powers, factorials, sums and products (also over i).)

PART III

9) (5p) Each of 10 white marbles, 10 black marbles and 10 marbles of other, different, colours is to be placed in one of three (large) boxes. The boxes are distinguishable, marbles of the same colour are considered identical and no box may be empty.
In how many ways can the marbles be distributed?
(Answers may contain integers, powers, factorials and the four elementary arithmetical operations.)

10) Let G be the full symmetry group of a cube (i.e. it contains all distance preserving transformations taking the cube to itself, not only rotations).
a) (1p) What is $|G|$, the order of G?
b) (1p) The vertices of the cube are numbered $1, 2, \ldots, 8$. How many elements of G correspond to odd permutations in S_8 (the permutations of $\{1, 2, \ldots, 8\}$)?
c) (3p) A colouring of the six edges of a regular tetrahedron is said to be **chiral** if the so coloured tetrahedron can not be rotated to look like its mirror image. How many such chiral colourings are there, if k colours can be used?



A tetrahedron

Good luck!

Suggested solutions will be posted on the course's web site.