Problems # 1 1973. MISCELLANEOUS

1. Let f be a positive continuous function on $[0,1]$.
Prove:

$$\exp\left(\int_0^1 \log f(x)\ dx\right) \quad \leq \int_0^1 f(x)\ dx.$$

(Can = occur ?)

2. Let a be an integer ≥ 0 and $0 < d$. Then $A(a,d)$ is the set (arithmetic progression) of integers $\{a, a+d, a+2d, ...\}$.
Prove, if $2 \leq d_1 < d_2 < ...d_r$ and $a_1, ..., a_r$ are arbitrary, that no disjoint union of the sets $A(a_i, d_i)(i = 1, 2, ..., r)$ can include all the positive integers (or even all sufficiently large integers).

3. Prove that the following geometric configuration exists:
Two planes Π_1 and Π_2 in 3-space, a point O not lying on either plane, and circles $C_1 \subset \Pi_1$, $C_2 \subset \Pi_2$ such that, under central projection from O, C_1 and C_2 are mapped on one another. Their centers however do not correspond.

4. Prove it is impossible, using only a ruler (in the "orthodox way"), to locate the center of a given circle.

5. Let $P(x,y)$ be a polynomial with real coefficients, which vanishes on the circle $x^2 + y^2 = 1$. Prove there exists a polynomial $Q(x,y)$ such that $P(x,y) = (x^2 + y^2 - 1)Q(x,y)$.

6. Let a_n denote the number of ways of placing n-pence postage on an envelope, using $1-, 2-, 3-$ and $4-$penny stamps pasted in a row. (The order between stamps of different values are taken into account).
What is $f(x) = \sum_0^\infty a_n x^n$?