Selected Problems – Set # 3
Inequalities, continued

(1) Find positive x, y, z such that $xy + yz + xz = 3$ and $x^2y^3z^4$ is maximum.

(2) *Prove*, for positive x, y: $(xy(x+y))^2 \leq \frac{4}{27}\left(x^2 + xy + y^2\right)^3$, and *deduce* that if a cubic equation $x^3 + ax + b = 0$ has three real unequal roots, then $\frac{b^2}{4} + \frac{a^3}{27} < 0$.

(3) *Prove*: $x^5 + ax^4 + 10x^3 + cx^2 + dx + e = 0$, with $|e| > 1$, can't have 5 real roots of like sign.

(4) Find a right circular cone of maximum lateral area inscribed in a sphere of radius a.

(5) Find a right circular cone of minimum volume circumscribed to a sphere of a radius a.

(6) If $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $0 \leq b_1 \leq b^2 \leq \cdots \leq b_n$, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a_i b_i$$

(7) *Prove* $2^{-x} + 2^{-(1/x)} \leq 1 \quad \text{for} \quad \text{x} > 0.$

(8) *Prove* for all $n = 0, 1, 2, \ldots$ and $x > 0$

$$\left| e^{-x} - \frac{1}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}} \right| < \frac{1}{2^n}.$$

(9) *Prove* for $\lambda > 0$

$$\int_0^{\pi/2} (\cos x)^\lambda dx < \sqrt{\frac{\pi}{2\lambda}}.$$

(10) If $a_1, \dots, a_n$ are ≥ 0, not all 0, then

$$\prod_{i=1}^{n}(1+a_i) \geq \left(1+\frac{\sum\limits_{i=1}^{n} a_i^2}{\sum\limits_{i=1}^{n} a_i}\right)^{\left(\frac{(\sum a_i)^2}{\sum a_i^2}\right)}$$

Note. If possible, solve ## 1, 2, 3, 4, 5, 6 *without use of the calculus*, i.e. without derivatives!