Selected Problems, Set # 4
(Functional Equations and Functional Inequalities)

(1) Prove the inequality of arithmetic and geometric means along the following lines. Define, for $a \geq 0$ and $n = 1, 2, \ldots$

$$f_n(a) = \max x_1 x_2 \ldots x_n; \quad x_1 \geq 0, \ldots, x_n \geq 0, \quad \sum_1^n x_i = a.$$

Show that (i) $f_n(a) = a^n \lambda_n$ depends only on n, and (ii) $f_n(a) = \max_{0 \leq t \leq a} t f_{n-1}(a-t)$. Use these relations to determine $f_n(a)$.

(2) Let $f(x)$ be a non-negative function defined for $x \in \mathbb{R}^+ = \{x : 0 \leq x < \infty\}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$. *Prove* $f(x) = cx$, where $c \geq 0$ is a constant. (*Note:* f is *not* a priori asumed to be continuous!). (*) Prove the same conclusion holds if f is *Lebesgue measurable.*

(3) If $f(x, y)$ denotes the area of a rectangle with sides x, y (where x, y are ≥ 0) then, from natural axioms about areas we should have

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y) \quad \text{(all variables are } \geq 0)$$
$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$$

Prove. the only non-negative functions f on $\mathbb{R}^+ \times \mathbb{R}^+$ which satisfy these equations (identically in x, y, x_1, x_2, y_1, y_2) are $f(x, y) = cxy$ (where $c \geq 0$ is a constant).

(4) Let p be a fixed number, $0 < p < 1$ and consider the "weighted mean" function $f(x, y) = (1-p)x + py$. Clearly f satisfies the functional equations

$$f(x+t, y+t) = f(x, y) + t \quad \forall x, y, t, \lambda \in \mathbb{R}$$
$$f(\lambda x, \lambda y) = \lambda f(x, y).$$

Prove that, conversely, the weighted mean is the only solution of these equations. (Can you give a generalization with n variables in place of x, y?)

(5) In the axiomatic theory of statics, one is led to the functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$.

Prove that the only continuous solutions of this equation (here x, y are any points in $\mathbb{R}$) are $f(x) \equiv 0$, $f(x) = \cos cx$, $f(x) = \cosh cx$.

(6) Find all triples of continuous functions f, g, h on $\mathbb{R}$ such that $f(x+y) = g(x) + h(y)$ for all $x, y \in \mathbb{R}$.

(7) Find the most general continuous solution of

$$f(x) + f(y) = f\left(\sqrt{x^2+y^2+1}\right) \quad , \quad \forall x, y \in \mathbb{R}.$$

(8) *Prove:* a polynomial $P(x)$ such that

$$(x-1)P\left(x^2-1\right) = xP(x)^2 + 2P(x) \quad (\forall x \in \mathbb{R})$$

vanishes identically.

(9) Let $f(x)$ be a real-valued function on $\mathbb{R}$ such that

$$|f(x+a) - 2f(x) + f(x-a)| \leq a \quad \forall x \in \mathbb{R}, \quad a > 0. \qquad (*)$$

If moreover f is integrable over each finite interval, prove that f is continuous.

(10) With same set-up as in # 9, but $a^{3/2}$ replacing a on the right side of (*), prove f has a continuous derivative.

(11) Let f be real-valued and continuous on $[0,1]$ and suppose for every $x \in (0,1)$ there exists $a = a(x)$ such that $0 \leq x - a$, $x + a \leq 1$ and $f(x) = \dfrac{1}{2a}\displaystyle\int_{x-a}^{x+a} f(t)dt$. *Prove* f is a linear function.

(12) *Prove:* If f is real-valued and continuous on $[a,b]$ and there is at least one point of every chord of the curve $y = f(x)$, besides the end-points of the curve, which lies above or on the curve, then every point of every chord lies above or on the curve (so that f is a convex function).

(13) Let f be real-valued on $\mathbb{R}$ with two continuous derivatives, moreover $|f(x)| \leq A$, $|f''(x)| \leq B$, $\forall x \in \mathbb{R}$. *Prove* $\sup_x |f'(x)| \leq 2A + \dfrac{B}{4}$ and $\sup_x |f'(x)| \leq \sqrt{2AB}$.