Selected Problems, Set # 6
(Measure theory, etc.)

(1) Let $E \subset \mathbb{R}$ have positive measure. *Prove* the set of $x \in \mathbb{R}$ representable in the form $x-y$ with $x, y \in E$ contains an interval about 0.

(2) Let $E \subset \mathbb{R}$ have positive measure, n any positive integer. *Prove* E contains n points in arithmetic progression (i.e. points $x_1, \ldots, x_n$ with $x_{i+1}-x_i$ all equal and positive, $i = 1, \ldots, n-1$).

(3) Let $E \subset \mathbb{R}^2$ have positive (planar) measure. *Prove* there exists an equilateral triangle with its vertices all in E. (Can you give a common generalization of # 2 and # 3?)

(4) Let $E \subset \mathbb{R}$ have positive measure, and $0 < \alpha < 1$. *Prove* there is an interval I such that the ratio of the measure of $E \cap I$ to that of I equals α.

(5) Let f be a real-valued function on $\mathbb{R}$ whose derivative f' exists everywhere. Suppose $f'(x_1) = \alpha < f'(x_2) = \beta$. *Prove* for every λ, $\alpha < \lambda < \beta$ $\exists x_3 : f'(x_1) = \lambda$.

(6) Suppose pairwise disjoint open circular discs are removed from a square, of radii $\{r_n\}_{n=1}^{\infty}$, such that the residual set has planar measure 0. *Prove* $\sum_1^{\infty} r_n = \infty$.

(7) Let $f(x)$ be a real-valued function on $[0, 1]$ of bounded variation, and continuous. Let $n(t)$, where $t \in \mathbb{R}$, denote the number of points where $f(x) = t$ (possibly $n(t) = +\infty$). *Prove* $\int_{-\infty}^{\infty} n(t)dt$ is finite, and equals the total variation of f.

(8) Let Γ be a rectifiable Jordan arc in the plane; the set of real numbers c such that the line $x = c$ has infinitely many points of intersection with Γ, has measure zero.

(9) Let E be the set of real numbers in $(0, 1)$ with non-terminating decimal expansions. Let S be a measurable subset of E such that, if $x \in S$, all numbers whose decimal expansion agrees with that of x from some point onwards also belong to S. *Prove* the measure of S is either 0 or 1.

(10) (Continuation) For almost all numbers in E, the block 2 3 1 7 8 0 0 5 occurs infinitely often in its decimal expansion.

(11) Let $1 \leq n_1 < n_2 < n_3 < \dots$ be integers. For almost every positive number x, the fractional part of the numbers $n_1 x, n_2 x, n_3 x, \dots$ are everywhere dense in $(0, 1)$.

(12) Let $f(t)$ be a continuous function on $[0, 1]$, complex valued, whose range is the whole closed unit disc $(= \{z : |z| \leq 1)$ ("Peano curve"). *Prove* f satisfies no Hölder condition of order $> 1/2$; that is $\nexists C > 0$, $\alpha > 1/2$ such that $|f(t_2) - f(t_1)| \leq C|t_2 - t_1|^\alpha$. (* Show that there does exist a "Peano" f which satisifes a Hölder condition of order $1/2$).

(13) *Prove* the f in # 12 is not of bounded variation.

(14) Let $\{I_n\}_{n=1}^\infty$ be any intervals whose union covers the Cantor set, and $\alpha = \dfrac{\log 2}{\log 3}$. *Prove* $\sum_{n=1}^\infty |I_n|^\alpha \geq 1$ (here $|I|$ denotes the length of I).

(15) Given $n^2 + 1$ distinct positive integers, written in a row. *Prove* there are $n+1$ of them which, in the order they are written, are a monotone sequence. (Not measure theory, just a cute little elementary problem to fill up the page.)