Problem73# 1.1 Solution A

We first recall the general inequality

$$\log y \ \le y - 1 \quad \text{for} \quad y > 0. \tag{1}$$

Indeed, $g(y) := y - 1 - \log y$ tends to $+\infty$ as y tends to either 0 or ∞, and its derivative vanishes only at one point of $(0, \infty)$, namely at $y = 1$, so g attains its minimum value 0 at $y = 1$, and nowhere else. Substituting $f(x)$ for y in (1) and integrating over $I := [0, 1]$ gives

$$\int \log f(x)\ dx \le \int f(x)\ dx - 1.$$

We may replace here f by tf where t is a positive parameter at our disposal:

$$\log t + \int \log f(x)\ dx \ \le t \int f(x)\ dx - 1,$$

or regrouping terms

$$\int \log f(x)\ dx \ \le t \int f(x)\ dx - \log t - 1.$$

The right hand member can be denoted $h(t) := At - \log t - 1$ where $A := \int f(x)\ dx$. By an almost identical argument to that used in proving (1), $h(t)$ attains a minimum value equal to $\log A$ on $(0, \infty)$, at $t = 1/A$, and only there. Hence

$$\int \log f(x)\ dx \ \le \log A = \log \int f(x)\ dx$$

and exponentiating gives the required inequality. Equality holds if, and only if f is constant. Indeed, "if" is obvious. And, if f is not constant, then at the first step when we wrote $\log f(x) \ \le f(x) - 1$ strict inequality would hold on a set of positive measure, and so between the integrals, etc.

Remark. This proof generalizes to the case where I is replaced by any measure space (X, m) with m a probability measure and f integrable with respect to m.

HSS