Problem73# 1.1 Solution C

Since f is continuous on $[0,1]$ it is Riemann integrable and we can therefore use Riemann sums to solve this problem.

Set $R_n = \frac{1}{n}\sum_{j=1}^{n} \log f(j/n)$ and note that

$G_n = \exp(R_n) = \left(\prod_{j=1}^{n} f(j/n)\right)^{1/n}$ is the geometric mean of the numbers $\{f(j/n)\}_{j=1}^{n}$.

The corresponding arithmetic mean is $A_n = \frac{1}{n}\sum_{j=1}^{n} f(j/n)$ and we have $G_n \le A_n$ with equality only when all $f(j/n)$ are equal.

Hence,

$$(1) \qquad \exp\left(\frac{1}{n}\sum_{j=1}^{n} \log f(j/n)\right) \le \frac{1}{n}\sum_{j=1}^{n} f(j/n).$$

Letting $n \to \infty$ the Riemann sums converge to their corresponding integrals and we get:

$$(2) \qquad \exp\left(\int_0^1 \log f(x)\ dx\right) \le \int_0^1 f(x)\ dx \qquad \text{as required.}$$

It can also be shown that equality in (2) implies that $f =$ constant. We refer to Solution 2 where essentially the same fact (f is constant a.e.) is proved from the weaker condition that f is measurable.

With the same arguments as in S2 it can be shown that if equality holds in (2) then $\int_{\Delta_i} f\ dx = \int_{\Delta_j} f\ dx$ where Δ_i and Δ_j are any couple of dyadic intervals (see S2) of the same length $1/2^n$. Since this holds for any n, f must be constant in view of the continuity.