Problem 73# 1.4 Solution

Consider now the problem, to locate, using ruler alone, the center of a given circle.
In order to make this problem meaningful, one must of course have formalized unambiguously exactly which procedures are permitted. (For example, it is *not* permitted to draw a tangent from a point P to a given circle by placing the ruler on point P and, keeping it in contact with P, rotating it until we observe that it is tangent to the circle.) Admittedly there is some arbitrariness in the formulation of what is permitted, but let us take for granted that this has been done. (The reader may wish to propose a list of allowable operations.)

The argument which follows is a *metamathematical* proof, of a kind which perhaps is new to the student. What is amazing about it is, it refutes any proposed procedure for locating the circle's center using the allowed operations, not on the basis of a detailed scrutiny of the procedure, but on general grounds based on the invariance concept. Even if the proposed construction involves thousands of steps, we are able to say with assurance, and a priori, that it does not work.

Suppose we are given a circle C in the plane p and a proposed algorithm for constructing its center (i.e. ultimately drawing two lines whose intersection shall be the center of C). The algorithm (construction) might for example begin with: "Choose a point A not on C and draw a line L through A meeting C in two points D, E. Now choose a point F on C distinct from D and E and draw lines connecting it to D and E. ...(and so forth) ".

By virtue of the preceding problem, we can find another plane , say p' distinct from p, and a point O not on either plane such that projecting p on p' via O maps C to a circle C' , and the center of C to a point which is not the center of C' .

Let us go through the proposed construction, transforming each step to the corresponding one in plane p'. If we denote generically by X' he transform of the point (or line) X we see that A' is not on C', and the line L' through A' meets C' in two points, which are D' and E' .

Continuing in this manner, we see that the whole construction leading ultimately to the point Z that is proposed as the center of circle C has a counterpart in the plane p' and leads to a point Z' , the image of Z under the projection through O. Now, by the previous problem, if Z really is the center of C, then Z' cannot be the center of C' . Thus, it cannot be the case that Z is the center of C and Z' is the center of C'.

So, either the proposed construction has failed, or else its "shadow" in the plane p' has failed. But, each of these has the same legitimacy, the "shadow" construction in p' follows exactly the same algorithm as the original one in p, and must be correct if the original one was. The contradiction is now apparent, and our conclusion is that the proposed construction is impossible.

Some points for thought and discussion: Suppose we had allowed the use of compass as well as ruler. Then the center really is constructible. What breaks down if we try to repeat the argument? Also, is there any chance to prove, along the lines of the above argument, the impossibility of angle trisection using ruler and compass?

HSS