Problem73# 1.5 Solution

There are so many different modes of attack that this problem is almost a mini-course in several complex variables.

Let me here pursue one line, based on Hilbert's Nullstellensatz.

One is tempted to conclude from the hypotheses "by the Nullstellensatz" that for some positive integer m, P^m is divisible by Q. From this and the easily checked fact that Q is irreducible over the complex field then follows the desired conclusion that Q divides P.

However, the hypothesis for the applicability of the Nullstellensatz is:

$$P \text{ vanishes on the } \textit{complex} \text{ zeros of } Q. \qquad \text{(HC)}$$

We are only given that

$$P \text{ vanishes on the } \textit{real} \text{ zeros of } Q. \qquad \text{(HR)}$$

So, how to deduce (HC) from (HR)? We present two ways:

Variant 1. (HR) implies $P(\cos t, \sin t) = 0$ for all real t.

This implies the same identity for *complex* t (why?) which in turn implies (HC) (fill in the details).

This method, although elegant, is somewhat special, based on our knowing a convenient parametrization of $\{Q = 0\}$. The following argument, although longer, applies in much more general situations.

Variant 2. First let us study a simpler analogous problem: try the case $Q = y$.

Thus, we are given that $P(x, 0)$ is identically 0, and hence all partial derivatives of P w.r.t. x vanishes at $(0, 0)$. Therefore in the (terminating) Taylor expansion of P, all terms of type x^k for $k = 0, 1, 2, \ldots$ disappear. Each surviving term has the factor y, and we're done (here it makes little difference whether x, y were real or complex variables).

Now, let's use this insight to tackle the original problem. Make a change of (complex!) variables

$$X = x,\, Y = x^2 + y^2 - 1. \qquad (*)$$

We will initially work locally for, say, (x, y) on a nbhd of $(0, 1)$. The Jacobian of the transformation is $2y$ which is not 0 at $(0, 1)$ so $(*)$ is a diffeomorphism of a nbhd of $(0, 1)$ on a nbhd of $(0, 0)$. In fact, by the implicit function theorem for holomorphic functions this is a biholomorphic map between complex nbhds of $(x, y) = (0, 1)$ and $(X, Y) = (0, 0)$. If we set $P(x, y) =: f(X, Y)$, f is now defined as a holomorphic function on a nbhd N of $(0, 0)$ in C^2, and it vanishes for $Y = 0$ in this nbhd We are almost in the situation of the earlier simpler problem, except now f is not a polynomial. But, going back to that proof, we see it would have worked as well if P had been not a polynomial but a convergent power series, and we infer that $f(X, Y) = Y g(X, Y)$ where g is some convergent power series on a nbhd of $(0, 0)$. Thus,

$$P(x, y) = \left(x^2 + y^2 - 1\right) g\left(x, x^2 + y^2 - 1\right)$$

and we conclude: $P(x, y) = x^2 + y^2 - 1$ times a convergent power series in (x, y) for (x, y) in some (complex, i.e. in C^2) nbhd of $(0, 1)$.

Consequently, P vanishes on all the *complex* zeros of Q in some nbhd of $(0, 1)$ in C^2. But, there was nothing whatever special about the point $(0, 1)$, we could as well have chosen *any* point (x, y) in C^2 such that $x^2 + y^2 - 1 = 0$, since at every point of this set at least one of the first order partial derivatives $2x, 2y$ of Q is not equal to 0.

HSS