Problem73# 1.6 Solution A

To simplify notations, let us present the solution when there are only two denominations of stamps, of values a pence and b pence, where a and b are distinct positive integers with $a < b$.

Consider first the case where ORDER MATTERS. Let $p(m, s)$ denote the number of ways in which we can build up the postage m using s stamps, each of which has denomination a or b. We will encapsulate these numbers by means of the formal power series

$$f_s(x) := \sum_{m=0}^{\infty} p(m, s)x^m$$

Clearly $f_0(x)$ is identically 1 , and $f_1(x) = x^a + x^b$.
Now, $f_2(x) = (x^a + x^b)(x^a + x^b)$, since expansion of the product gives x raised to the exponents $a + a, a + b, b + a, b + b$ which signifies that postages achievable with two stamps are $2a$ (in one way) , $a + b$ (in two ways) , and $2b$ (in one way).
Similarly, $f_3(x) = (x^a + x^b)(x^a + x^b)(x^a + x^b)$, and so forth.

Therefore, the number $p(m)$ of ways to build up the postage m with SOME assemblage of stamps (order being taken into account) is the coefficient of x^m in the formal power series
$f_0(x) + f_1(x) + f_2(x) + ... = 1 + (x^a + x^b) + (x^a + x^b)^2 + ... = 1/(1 - x^a - x^b)$.
(Note that the series converges for x of sufficiently small absolute value.)

Consider now the case when order is not taken into account. Clearly , the number of ways to assemble the postage "m pence" is the number $q(m)$ of distinct solutions in non-negative integers t, u to the equation $ta + ub = m$.
Then

$$\sum_{m=0}^{\infty} q(m)x^m = \sum_{t=0}^{\infty} x^{at} \cdot \sum_{u=0}^{\infty} x^{bu} = \frac{1}{(1 - x^a)(1 - x^b)}.$$

So, we conclude that:
$q(m)$ equals the coefficient of x^m in the Taylor expansion of $\frac{1}{(1-x^a)(1-x^b)}$.

Remarks.
One can obtain good asymptotics for $p(m), q(m)$ by developing the respective rational generating functions $\frac{1}{(1-x^a-x^b)}$ and $\frac{1}{(1-x^a)(1-x^b)}$ in partial fractions. Notice that the first has a pole at the point x_0 which is the positive root of the equation $x^a+x^b=1$, and $0<x_0<1$, whereas the second has no poles in the open unit disk, but has poles on the unit circle. This reflects the fact that $p(m)$ is more rapidly growing as $m\to\infty$ than $q(m)$, indeed the former grow exponentially while the latter do not.Notice how this reflects our intuition that there are a great many more ways to achieve the postage m when order is taken into account, than when it is not.

Finally, to return to the original problem (with 4 kinds of stamps) the respective generating functions are

$\frac{1}{(1-x-x^2-x^3-x^4)}$ with order taken into account, and

$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ with order not taken into account.