Problem73# 3.10 Solution A

Set $A = \sum_{j=1}^{n} a_j$ and $B = \sum_{j=1}^{n} a_j^2$.

We want to show $\sum_{j=1}^{n} \log(1 + a_j) \geq \frac{A^2}{B} \log(1 + \frac{A}{B})$.

Use "method of the ghostly parameter".

Define $\varphi(t) = \sum_{j=1}^{n} \log(1 + a_j t) - \frac{A^2}{B} \log(1 + \frac{Bt}{A})$.

We have to prove $\varphi(1) \geq 0$.

Now, $\varphi(0) = 0$, and we are done if we can establish $\varphi'(t) \geq 0, \quad t > 0$.

But, $\varphi'(t) = \sum_{j=1}^{n} \frac{a_j}{1 + a_j t} - \frac{A^2}{B} \left(\frac{1}{1 + \frac{Bt}{A}} \right) \cdot \frac{B}{A} =$

$\sum_{j=1}^{n} \frac{a_j}{1 + a_j t} - \frac{A^2}{A + Bt}$ and we have to show

$$A^2 \leq (A + Bt) \cdot \sum_{j=1}^{n} \frac{a_j}{1 + a_j t} \quad \text{or} \quad \left(\sum_{j=1}^{n} a_j \right)^2 \leq \sum_{j=1}^{n} a_j (1 + a_j t) \cdot \sum_{j=1}^{n} \frac{a_j}{1 + a_j t}$$

which is just Cauchy's inequality, namely

$$\left(\sum_{j=1}^{n} a_j \right)^2 = \left(\sum_{j=1}^{n} (\sqrt{a_j} \sqrt{1 + a_j t}) \left(\frac{\sqrt{a_j}}{\sqrt{1 + a_j t}} \right) \right)^2 \leq$$

$$\leq \sum_{j=1}^{n} a_j (1 + a_j) \cdot \sum_{j=1}^{n} \frac{a_j}{1 + a_j t} \ . \qquad \text{QED.}$$

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