Problem73# 3.2 Solution

Problem 2. Prove, for positive x, y

$$[xy(x+y)]^2 \leq \left(\frac{4}{27}\right)\left(x^2+xy+y^2\right)^3. \tag{1}$$

Proof. Since both left and right hand expressions are homogeneous of degree 6, we may assume without loss of generality that

$$x^2+xy+y^2=3 \tag{2}$$

and have to deduce (3) $xy(x+y) \leq 2$.

By the inequality of the means, the geometric mean of the three summands in the left member of (2), which is $[(x^2)(xy)(y^2)]^{(1/3)} = xy$ does not exceed 1. Denote $t := xy$. Then (3) follows if we show $(xy)^2(x^2+2xy+y^2) \leq 4$, i.e. $t(t+3) \leq 4$. But $t \leq 1$, so the last inequality is clearly true since $t(t+3)$ increases with t for positive t, QED.

The other clause in the problem, about roots of a cubic, is obvious and I won't include the proof here.