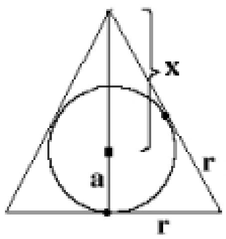
Problem73# 3.5 Solution



With notation from the figure we get:
Cone height, $h = a + x$.
Cone base radius r satisfies
$r^2 + (x+a)^2 = (r + \sqrt{x^2 - a^2})^2$
i.e. $2xa + 2a^2 = 2r\sqrt{x^2 - a^2}$,
$r = \dfrac{a(x+a)}{\sqrt{x^2 - a^2}} = a\sqrt{\dfrac{x+a}{x-a}}$

Cone volume $V = \dfrac{\pi r^2 h}{3} = \dfrac{\pi}{3} a^2 \dfrac{x+a}{x-a}(x+a) = \dfrac{\pi a^2}{3} \cdot \dfrac{(x+a)^2}{x-a}$.
To minimize V without calculus divide and use the AG inequality (case $n = 2$):

$$\frac{(x+a)^2}{x-a} = x + 3a + \frac{4a^2}{x-a} = 4a + x - a + \frac{4a^2}{x-a} = 4a + 2 \cdot \tfrac{1}{2}\left(x - a + \frac{4a^2}{x-a}\right) \geq$$

$$4a + 2\sqrt{(x-a)\frac{4a^2}{(x-a)}} = 4a + 2 \cdot \sqrt{4a^2} = 8a.$$

The volume is clearly minimal when the terms of the arithmetic mean are equal, i.e. when $x - a = \dfrac{4a^2}{x-a} \quad \Leftrightarrow \quad x = 3a. \quad$ (since $x > a$).
Hense, the circumscribed cone has minimal volume when its height is 4a.

BE/GJ