Problem 73 # 3.7 Solution

Problem 7: Prove

(*) $\qquad 2^{-x} + 2^{-1/x} \leq 1$ for $x > 0$.

Set $f(x) = 2^{-x} + 2^{-1/x}$. Since $f(x) = f(1/x)$ it is sufficient to prove (*) for $0 < x \leq 1$.

Observe $f(1) = \lim_{x \to 0+} f(x) = 0$.

Let $M = \max f(x), \quad 0 \leq x \leq 1$

(f assumed continously extended to $x = 0$).

If $M > 1$ then M is attained at a point in $(0, 1)$ where $f'(x)$ vanishes.

Now $f'(x) = 2^{-x}(-\log 2) + 2^{-1/x} \cdot \log 2/x^2$.

So $f'(\xi) = 0 \Rightarrow 2^{-\xi} = 2^{-1/\xi} \cdot \dfrac{1}{\xi^2}$

and at such a point, $f(\xi) = 2^{-\xi} + 2^{-\xi} \cdot \xi^2$.

Thus it is sufficient to prove $2^{-\xi}(1 + \xi^2) \leq 1$ for $0 < \xi < 1$.

i.e. $\varphi(\xi) = 2^{\xi} - \xi^2 - 1 \geq 0$ on $(0, 1)$,

But $\varphi''(\xi) = 2^{\xi}(\log 2)^2 - 2 \leq 2((\log 2)^2 - 1) < 0$.

Hence φ is concave so, since $\varphi(0) = \varphi(1) = 0$, we get $\varphi(\xi) \geq 0$ on $(0, 1)$.

QED.

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