Problem73# 3.8 Solution

Set $s_n(x) = 1 + x + ...\frac{x^n}{n!}$.

We want to prove that $M_n = \max_{0<x} |\Delta_n(x)| = \max_{0<x} \left| \frac{1}{s_n(x)} - e^{-x} \right|$

satisfies $\quad M_n \leq 2^{-n}$

Let ξ_n be the point where $\Delta_n(x)$ attains its maximum, $\Delta_n(\xi_n) = M_n$.
(note that $\Delta_n(0) = \Delta_n(\infty) = 0$ and $\Delta_n(x) > 0$ in the interior.)
Then $\Delta_n'(\xi_n) = 0$,
and $-\frac{s_n'}{s_n^2} + e^{-x} = 0$ at $x = \xi_n$, $\quad$ i.e. (since $s_n' = s_{n-1}$)

(1) $\qquad \frac{s_{n-1}(\xi_n)}{s_n(\xi_n)^2} = e^{-\xi_n}$,

Hence $M_n = \frac{1}{s_n(\xi_n)} - e^{-\xi_n} = \frac{1}{s_n(\xi_n)} - \frac{s_{n-1}(\xi_n)}{s_n(\xi_n)^2} = \frac{s_n(\xi_n) - s_{n-1}(\xi_n)}{s_n(\xi_n)^2} = \frac{\xi_n^n/n!}{s_n(\xi_n)^2}$

But $s_n(\xi_n)^2$ must be greater than its ξ_n^n-term which must coincide with the ξ_n^n-term of the Taylor expansion of $e^{\xi_n^n} \cdot e^{\xi_n^n} = e^{2\xi_n}$, i.e. $\frac{2^n \xi_n^n}{n!}$.

And so $M_n = \frac{\xi_n^n/n!}{s_n(\xi_n)^2} \leq \frac{\xi_n^n/n!}{2^n \xi_n^n/n!} = \frac{1}{2^n}$. QED

MB/HSS/TJ