Problem 73# 3.9 Solution

Problem 9. The key idea is to prove $\cos x \le \exp\left(\frac{-x^2}{2}\right)$ for x on the interval $[0, \pi/2]$, then replace cos in the integral by $\exp\left(\frac{-x^2}{2}\right)$, and moreover extend the integration over $(0, \infty)$ to get the indicated upper bound. So, the crucial verification is that

$$F(x) := (\cos x)\left(\exp\left(x^2 2\right)\right) \text{ is } \le 1 \text{ for } x \text{ in } \left[0, \frac{\pi}{2}\right].$$

Now, $F(0) = 1$, and $F'(x) = \exp\left(\frac{x^2}{2}\right)[-\sin x + x\cos x]$, so if we establish that $x\cos x < \sin x$ on $(0, \frac{\pi}{2})$ we'll know F is decreasing on this interval, and we're done. The rest is easy calculus. Here is one variant: we want to show $x < \tan x$ or $\arctan x < x$, that is $\int_0^x \frac{dt}{(1+t^2)} < x$ for $x > 0$, which is obvious.

HSS