LECTURES ON BALAYAGE

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ABSTRACT. We give an exposé over some recent developments in potential theory centred around the notion of partial balayage.

Partial balayage means balayage (sweeping of measures) to a prescribed density, which then is attained on a set which is not known in advance. This gives a free boundary problem of obstacle type. Performed continuously in time partial balayage is equivalent to moving boundary problems for Hele-Shaw flows, and in the backward direction (inverse balayage) it leads in the limit to notions of potential theoretic skeletons (sometimes called mother bodies).

1. INTRODUCTION

We shall discuss questions of balayage in Newtonian potential theory. The Newton kernel in $\mathbb{R}^n$ $(n \geq 2)$ is

$$U(x) = \begin{cases} c_2 \log \frac{1}{|x|} & (n = 2), \\ \frac{c_n}{|x|^{n-2}} & (n \geq 3), \end{cases}$$

the constant $c_n > 0$ chosen so that $-\Delta U = \delta$ (the Dirac measure at the origin). If μ is a positive measure with compact support its Newtonian potential is the convolution

$$U^\mu = U * \mu.$$

The corresponding field is the gradient ∇U^μ and the energy is

$$\|\mu\|_e^2 = \iint U(x-y)d\mu(x)d\mu(y) = \int U^\mu d\mu.$$

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Regarding this as a squared norm (when $n \geq 3$) there is a corresponding inner product, namely the mutual energy between the two mass distributions:

$$(\mu, \nu)_e = \int U^\mu d\nu = \int U^\nu d\mu.$$

The definition of U^μ makes sense also for signed measures (charges) and for more general distributions (with compact support, e.g.). When $n = 2$, the "norm" $\|\mu\|_e^2$ can become negative, but it is positive for signed measures with zero net mass. Indeed, for such a measure, as well as for any measure in higher dimensions, the potential U^μ decays at infinity in such a way that the Green formula can be applied to give

$$\|\mu\|_e^2 = \int_{\mathbb{R}^n} |\nabla U^\mu|^2 dm.$$

Here m denotes Lebesgue measure.

By a measure (or mass distribution) we shall generally mean a positive Borel measure which is finite on compact sets. Thus, from another point of view, a measure is the same thing as a positive distribution. A signed measure is simply the difference between two (positive) measures.

Some general references in potential theory, suitable for these notes, are [68] [57] [76] [24], [54], [6].

2. Classical balayage

If a measure μ is changed in some way its potential U^μ will also change, at least somewhere, because μ can be recovered from U^μ via

$$-\Delta U^\mu = \mu.$$

However, it is possible to redistribute μ in such a way that U^μ remains unchanged in part of the space, e.g. outside a given domain D. This is where the notion of balayage (or sweeping) comes in.

Indeed, **classical balayage** of a measure μ with respect to a bounded domain $D \subset \mathbb{R}^n$ is the process of cleaning D from any mass of μ in such a way that the potential remains unchanged outside D. We shall use the notation

$$\nu = \mathrm{Bal}\,(\mu, D^c)$$

for this process $\mu \mapsto \nu$ and the requirements on ν are thus that

$$\nu = 0 \text{ in } D, \tag{2.1}$$

$$U^\nu = U^\mu \text{ outside } D. \tag{2.2}$$

By the maximum principle the inequality

$$U^\nu \leq U^\mu \text{ in } \mathbb{R}^n \tag{2.3}$$

automatically holds. It is allowed that μ has mass also outside D. That part of μ will simply be unchanged, while $\mu|_D$ will redistribute on ∂D. For simplicity we assume in this section some mild regularity of D (that it is

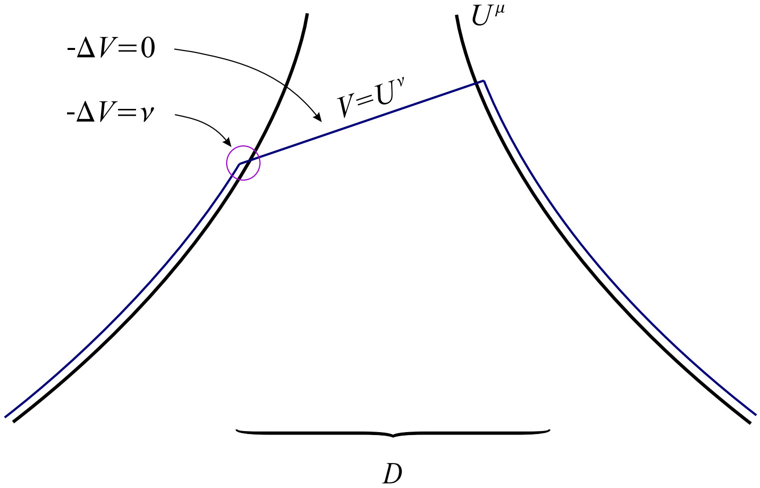

Figure 1: Classical balayage and the Dirichlet problem.

regular for the Dirichlet problem), otherwise one must allow for a small exceptional set in (2.2).

The idea of balayage goes back at least to C.F. Gauss [34] (see [24], p.799f, for historical accounts in general). The physically most intuitive way of producing ν from μ is by minimizing the energy for the change:

$$\text{Min } \|\mu - \nu\|_e^2 : \quad \nu = 0 \text{ in } D. \tag{2.4}$$

In two dimensions one should add the side condition that $\int d\nu = \int d\mu$.

The above approach was made rigorous by O. Frostman [32], E. Cartan [17] and others: there exists a unique minimizer ν and this has the properties (2.1), (2.2). The fact that not all measures have finite energy does not cause any problem because the energy norm may be decomposed as

$$\|\mu - \nu\|_e^2 = \|\mu\|_e^2 - 2(\mu, \nu)_e + \|\nu\|_e^2$$

and if μ (or rather $\mu|_D$) has infinite energy one simply drops the term $\|\mu\|_e^2$ (or $\|\mu|_D\|_e^2$) and minimizes the rest, which can always be given a meaning.

Another way of constructing the balayage measure is by solving a Dirichlet problem: let V solve

$$\begin{cases} -\Delta V = 0 & \text{in } D, \\ V = U^\mu & \text{on } \partial D \end{cases} \tag{2.5}$$

and extend V by $V = U^\mu$ outside D. Then V will be the potential of a measure ν, $V = U^\nu$, and by construction that measure has the properties (2.1), (2.2), so it is $\text{Bal}\,(\mu, D^c)$. See Figure 1.

In terms of the difference $u = U^\mu - V = U^\mu - U^\nu$ we have $\Delta u = \nu - \mu$, hence

$$\begin{cases} -\Delta u = \mu & \text{in } D, \\ \quad u = 0 & \text{outside } D, \end{cases}$$

and we may write

$$\mathrm{Bal}\,(\mu, D^c) = \mu + \Delta u.$$

In case μ is Dirac measure δ_a at a point $a \in D$ the above function u will be the Green function of D with pole at a,

$$u = G_D(\cdot, a).$$

In this case

$$\omega_a = \mathrm{Bal}\,(\delta_a, D^c)$$

is known as **harmonic measure** of ∂D (with respect to a). Expressed directly in terms of $G_D(\cdot, a)$ it is

$$d\omega_a = -\frac{\partial G_D(\cdot, a)}{\partial n} ds, \tag{2.6}$$

where $\frac{\partial}{\partial n}$ denotes outward normal derivative and ds is hypersurface measure on ∂D.

In the special case that D is a ball the solution of the Dirichlet problem (2.5) is fully explicit ($V|_D$ is the Poisson integral of $U^\mu|_{\partial D}$) and V is referred to as the **Poisson modification** of U^μ. For a general D, the solution of (2.5) may be obtained as a limit of successive Poisson modifications in balls contained in D, starting from any superharmonic ($-\Delta V \geq 0$) function which equals U^μ on ∂D. For each Poisson modification in a ball B the function V decreases and becomes more harmonic. Indeed, the mass $-\Delta V|_B$ is swept to ∂B. The fact that the potential decreases is a guarantee that a definite progress is made at each step. Thus the whole process converges, and in the limit all mass is swept to ∂D and the function is harmonic in D.

The above method of solving the Dirichlet problem, by successive sweeping operations, was invented by H. Poincaré [83], [84]. O. Perron [82] realized that the final result simply is the infimum of all superharmonic functions in D having the prescribed (or larger) boundary values on ∂D, and the method often bears his name, **Perron's method**. See also [113].

In the context of balayage Perron's observation shows that the potential U^ν of $\nu = \mathrm{Bal}\,(\mu, D^c)$ is the *smallest* of all functions V satisfying

$$\begin{cases} V \geq U^\mu & \text{outside } D, \\ -\Delta V \geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Hence U^ν agrees with what is called the **reduced function** of U^μ over D^c in D (cf. Figure 1). One standard notation for this is

$$U^\nu = \hat{R}^{D^c}_{U^\mu}.$$

The hat on R indicates that the function has been normalized to be lower semicontinuous (as a superharmonic function should be). The reduction operation was much developed by M. Brelot [9], [10], and it is one of the basic tools for balayage.

A way to rephrase the reduction idea is to say that that U^ν solves an **obstacle problem** [72], [88]: U^ν is the smallest superharmonic function passing above the obstacle ψ defined by $\psi = U^\mu$ on D^c, $\psi = -\infty$ in D. There is also another obstacle type problem which produces U^ν, this time from below, and which is more appropriate for the generalizations we wish to make in the next section. This is that U^ν is the *largest* of all functions V satisfying

$$\begin{cases} V \leq U^\mu & \text{in } \mathbb{R}^n, \\ -\Delta V \leq 0 & \text{in } D. \end{cases} \tag{2.7}$$

It is not hard to verify this statement, which also conforms with one of the standard ways of introducing the Green function $G_D(\cdot, a)$, namely as U^{δ_a} minus the largest subharmonic function V in D satisfying $V \leq U^{\delta_a}$.

If one wants to extend classical balayage to signed measures there are two possibilities. One is to keep the requirements (2.1), (2.2) as they are. This leads to a linear map $\mu \mapsto \nu$, which can be viewed as the adjoint of the operator (the "Dirichlet solver") which extend continuous functions on ∂D to functions continuous on $\overline{D}$, harmonic in D. See for example [89] (section 5.22 there) and [108] for elegant treatments. (In [108] this point of view is developed further.) For this linear balayage (2.3) fails in general.

The other possibility is to relax (2.1) to $\nu \leq 0$ in D. With this as the side condition in (2.4) one gets a map $\mu \mapsto \nu$ which is an orthogonal projection onto a convex cone. Then (2.3) remains valid. It is this approach to balayage that we shall keep and develop in these notes.

A variant of classical balayage, with reversed geometry, is the **equilibrium distribution**. Here $K = \mathbb{R}^n \setminus D$ is compact, and given a number $\alpha > 0$ one minimizes

$$\text{Min}\, \|\nu\|_e^2 : \quad \nu = 0 \text{ in } D, \int d\nu = \alpha.$$

One may think of this as the balayage of a point mass of strength α placed at infinity and denote the result

$$\nu = \text{Bal}\,(\alpha\delta_\infty, K). \tag{2.8}$$

Here δ_∞ has only a symbolic meaning, at least in dimension $n \geq 3$.

The measure ν has the equilibrium property that

$$U^\nu = \beta \text{ on } K$$

for some constant β which depends (linearly) on α. The energy is $\|\nu\|_e^2 = \alpha\beta$.

In the electrostatic interpretation, K is a perfect conductor and α is the charge on it, which distributes to have constant ($= \beta$) potential on K. The **capacity** of K (with respect to infinity) then is

$$\text{Cap}\,(K) = \frac{\alpha}{\beta}, \tag{2.9}$$

at least in dimension $n \geq 3$.

In two dimensions there is the disadvantage that β may become zero or negative and one usually considers the **logarithmic capacity** instead. In terms of our notations this is

$$\operatorname{Logcap}(K) = \exp[-\frac{2\pi\beta}{\alpha}].$$

In physics one really only considers the capacity between *two* conductors. In dimension $n \geq 3$ there is a good limit if one of them is moved to infinity, but not so in two dimensions.

3. Partial balayage

The word "balayage" means sweeping, or clearing dust away (with e.g. a brush or broom) in French. In classical balayage one "cleans" a domain D completely from any mass sitting there. **Partial balayage** means that one only makes some partial cleaning. The role of the domain D is then taken over by a measure λ which tells how much mass (or "dust") is allowed to be left. Partial balayage of μ to λ will be denoted $\operatorname{Bal}(\mu, \lambda)$.

For classical balayage with respect to D, the "measure" (it is not a measure in our previous sense) λ is

$$\lambda(E) = \begin{cases} 0 & \text{if } E \subset D, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Thus, in the notation for classical balayage, $D^c = \mathbb{R}^n \setminus D$ should be interpreted as the "measure" which is zero in D, plus infinity on D^c.

The emphasize of partial balayage as outlined in these notes will rather be on measures λ similar to Lebesgue measure. Indeed, $\lambda = m$ is a case of major interest, and in general we shall assume that λ is absolutely continuous with respect to Lebesgue measure,

$$\lambda = \rho m, \tag{3.2}$$

where the density function ρ satisfies something like

$$0 < c_1 \leq \rho \leq c_2 < \infty \tag{3.3}$$

(c_1, c_2 constants).

It takes some extra efforts to incorporate measures like (3.1) into the picture, and we shall not take these efforts here (it is done in [46]). On the other hand, the assumption (3.2), (3.3) can be considerably relaxed without any additional labor. For example, the lower bound in (3.3) is really needed only in a neighbourhood of infinity.

The definition of partial balayage is intuitive and simple. The fixed data are only the measure λ satisfying (3.2), (3.3) (e.g.) and the balayage process $\mu \mapsto \nu$ may be defined by the requirement that μ (a measure with compact support) is to be replaced by a measure $\leq \lambda$ using as little work (energy) as possible. Thus we have

Definition 3.1. *Partial balayage of μ to λ,*

$$\nu = \mathrm{Bal}\,(\mu, \lambda),$$

is the unique solution of

$$\mathrm{Min}\,\|\mu - \nu\|_e^2 : \quad \nu \leq \lambda. \tag{3.4}$$

If μ happens to have infinite energy one minimizes $-2(\mu, \nu)_e + \|\nu\|_e^2$ instead. In two dimensions, one adds the side condition $\int d\nu = \int d\mu$.

In (3.4) it is understood that ν ranges over (positive) measures. However, it is actually more convenient to let ν range over all signed measures satisfying $\nu \leq \lambda$, the minimizer still turns out to be positive (see (3.12)).

Allowing thus signed measures the side condition $\nu \leq \lambda$ defines a convex cone which can be shown to be complete with respect to $\|\cdot\|_e$. It follows that there exists a unique minimizer ν and that this is characterized by the variational condition

$$(\mu - \nu, \nu - \sigma)_e \geq 0 \text{ for all } \sigma \leq \lambda.$$

(In two dimensions one only varies over σ with $\int d\sigma = \int d\mu$. As an alternative in two dimensions one may work in a large bounded domain and use the Green potential for that instead of the Newton potential, in order to avoid the inconveniences caused by the special properties of the logarithmic kernel.)

We may write the above variational condition as

$$\int (U^\mu - U^\nu) d(\nu - \sigma) \geq 0 \text{ for all } \sigma \leq \lambda. \tag{3.5}$$

Since $\nu \leq \lambda$, any $\sigma \leq \nu$ is allowed in (3.5) showing that

$$U^\mu - U^\nu \geq 0.$$

Next, choosing $\sigma = \lambda$ (or rather σ with compact support approximating this) gives

$$\int (U^\mu - U^\nu) d(\nu - \lambda) \geq 0,$$

which combined with the previous inequality and $\nu \leq \lambda$ shows that

$$\int (U^\mu - U^\nu) d(\nu - \lambda) = 0.$$

In summary, $\nu = \mathrm{Bal}\,(\mu, \lambda)$ satisfies

$$\nu \leq \lambda, \tag{3.6}$$

$$U^\nu \leq U^\mu, \tag{3.7}$$

$$\int (U^\mu - U^\nu) d(\lambda - \nu) = 0. \tag{3.8}$$

These conditions in fact characterize ν since, in the other direction, they imply that for any $\sigma \le \lambda$,

$$\int (U^\mu - U^\nu) d(\nu - \sigma) = \int (U^\mu - U^\nu) d(\nu - \lambda) + \int (U^\mu - U^\nu) d(\lambda - \sigma) \ge 0,$$

i.e., that (3.5) holds.

Another way to define $\mathrm{Bal}\,(\mu, \lambda)$ is the following, which generalizes (2.7).

Definition 3.2. *Partial balayage of μ to λ is the measure*

$$\mathrm{Bal}\,(\mu, \lambda) = -\Delta V^\mu,$$

where V^μ is the largest of all functions (or even distributions) V satisfying

$$V \le U^\mu \text{ in } \mathbb{R}^n,$$

$$-\Delta V \le \lambda \text{ in } \mathbb{R}^n.$$

See Figure 2 below. It is not hard to show that such a largest function exists. If for example $\lambda = m$ one may add $\frac{1}{2n}|x|^2$ to V and the task becomes that of finding the largest subharmonic function $V(x) + \frac{1}{2n}|x|^2$ which is $\le U^\mu(x) + \frac{1}{2n}|x|^2$. By standard results in potential theory [24], [6] such a largest function exists. Alternatively, by turning the picture upside-down we have an obstacle problem of standard form, which is known to have a unique solution.

It is easy to see (for this the lower bound in (3.3) is needed, at least far away) that $V^\mu = U^\mu$ outside a compact set, hence V^μ has the behaviour of a potential at infinity. Thus

$$V^\mu = U^\nu,$$

where $\nu = \mathrm{Bal}\,(\mu, \lambda)$ with the new definition.

Now we must of course show that the two definitions of $\mathrm{Bal}\,(\mu, \lambda)$ are the same. For this we adopt the second definition and show that it satisfies (3.6), (3.7), (3.8). We first introduce the **saturated set** for $\nu = \mathrm{Bal}\,(\mu, \lambda)$. It is defined by

$$\Omega = \Omega(\mu) = (\text{the largest open set in which } \nu = \lambda)$$

$$= \mathbb{R}^n \setminus \mathrm{supp}\,(\lambda - \nu).$$

A more complete notation would be $\Omega(\mu, \lambda)$.

If at some point $x \in \mathbb{R}^n$ we have $V^\mu < U^\mu$, then this strict inequality persists in a whole neighbourhood of x because U^μ is lower semicontinuous and V^μ is upper semicontinuous. Therefore there is room to make a Poisson type modification of V^μ in a small ball B around x: one replaces V^μ in B by a function V satisfying $-\Delta V = \lambda$ in B, $V = V^\mu$ on ∂B. This will make V^μ larger, unless it already satisfies $-\Delta V^\mu = \lambda$ in B.

Since V^μ was defined to be largest possible we conclude that

$$\{V^\mu < U^\mu\} \subset \Omega(\mu). \tag{3.9}$$

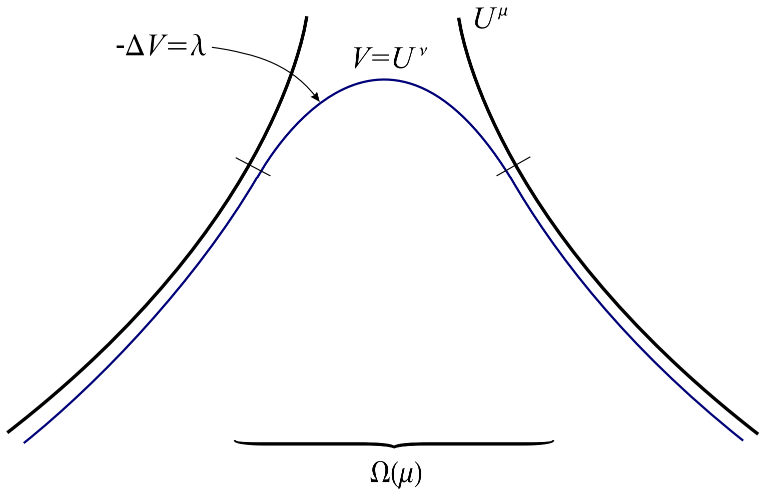

Figure 2: Partial balayage in terms of potentials.

In view of $V^\mu = U^\nu$ and the definition of $\Omega(\mu)$ this implies (3.8). Since (3.6), (3.7) are clear by construction we have now shown that the two definitions of $\mathrm{Bal}\,(\mu,\lambda)$ are equivalent.

One may notice that (3.9) can be written

$$U^\nu = U^\mu \text{ outside } \Omega(\mu)$$

and then says that the balayage measure ν is graviequivalent to μ outside $\Omega(\mu)$. In summary, partial balayage of μ produces a measure ν and an open set $\Omega = \Omega(\mu)$ such that ν has the prescribed density ($\nu = \lambda$) inside Ω and the prescribed potential ($U^\nu = U^\mu$) outside Ω.

The simplest example of partial balayage is this.

Example. Take $\lambda = m$, $\mu = \delta_a$ for some point $a \in \mathbb{R}^n$. Then

$$\Omega(\mu) = B(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\},$$

with r chosen so that $|B(a,r)| = \int d\mu = 1$, and

$$\mathrm{Bal}\,(\delta_a, m) = m|_{B(a,r)}.$$

Figure 2 illustrates exactly this case.

In general, the structure of $\mathrm{Bal}\,(\mu,\lambda)$ will be

$$\mathrm{Bal}\,(\mu,\lambda) = \lambda|_{\Omega(\mu)} + \text{ remainder}, \tag{3.10}$$

where the remainder is usually undesired and in any case is a measure $\leq \lambda$ with support outside $\Omega(\mu)$. At least under some regularity assumptions, e.g. that μ has a density in L^∞, the formula is more precisely

$$\mathrm{Bal}\,(\mu,\lambda) = \lambda|_{\Omega(\mu)} + \mu|_{\Omega(\mu)^c}, \tag{3.11}$$

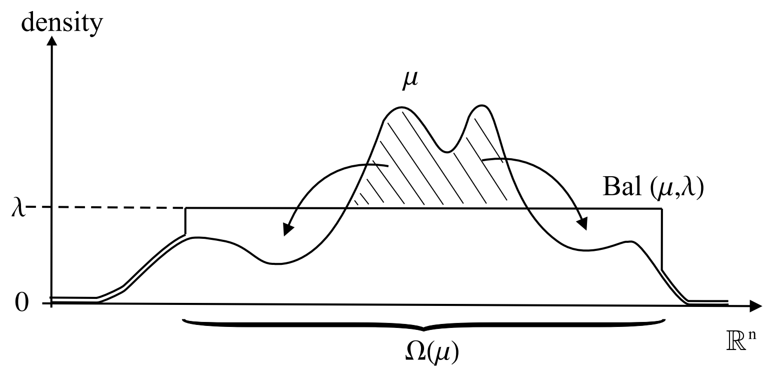

Figure 3: Structure of partial balayage.

saying that μ is left untouched outside $\Omega(\mu)$. See Figure 3.

NOTATIONAL REMARK. When writing e.g. $\lambda|_\Omega$, the restriction of the measure λ to the set Ω, it is understood that $\lambda|_\Omega$ is automatically extended by zero outside Ω.

One way to prove (3.11) is to first prove the estimate

$$\min(\mu, \lambda) \leq \text{Bal}(\mu, \lambda) \leq \lambda, \tag{3.12}$$

which always holds. It follows that if $\mu \in L^\infty$ then also $\nu \in L^\infty$, and then standard results from Sobolev theory [72] show that almost everywhere on the coincidence set $\{U^\nu = U^\mu\}$, in particular outside $\Omega(\mu)$, we have $\Delta U^\nu = \Delta U^\mu$, i.e. $\nu = \mu$. This gives (3.11).

Finally, we mention that it is always possible to perform balayage in smaller steps. For example, if $\lambda_1 \leq \lambda_2 + \mu_2$ we have

$$\text{Bal}(\mu_1 + \mu_2, \lambda_1) = \text{Bal}(\text{Bal}(\mu_1, \lambda_2) + \mu_2, \lambda_1). \tag{3.13}$$

In the same vein, partial balayage can be performed as a continuous process, say with μ replaced by a monotone family $\mu(t)$ where t is a time parameter, and then (3.13) give natural semigroup properties. Continuous partial balayage will be discussed in Section 6.

HISTORICAL NOTE. The idea of partial balayage goes back at least to the work of the Bulgarian geophysicist D. Zidarov who, probably together with some of his colleagues (e.g. Z. Zhelev), introduced an intuitive and numerical version of what he called "gravi-equivalent mass scattering" or "bubbling". See [119] and references therein. The ideas were further developed by O. Kounchev [73], [74] and others.

Later, and independently, M. Sakai developed an intricate and very precise method for constructing certain kinds of quadrature domains, and this exactly amounts to partial balayage [95]. Shortly afterwards and in parallel with developments in general free boundary theory more streamlined methods, using e.g. variational inequalities, were found for constructions equivalent to partial balayage [25], [96], [22], [98], [38], [39], [41]. See also [26], [30]. Further developments and generalizations were made in [46], [106], [58], [50], [66], [42] (survey), to mention just a few sources. The presentation in this note follows essentially [46], which contains full details on many of the topics slipped over here. For Section 7, [43] is the main reference.

One of the main inputs to the theory has all the time been extremal and other problems in complex analysis giving rise to domains having graviequivalence properties similar to those obtained with partial balayage. In [3], [4] D. Aharonov and H.S. Shapiro introduced the name quadrature domain for some classes of such domains, and they have been systematically studied since that time. See e.g. [5], [90], [91], [92], [93], [21], [107], [85], [79], [52], [18].

As will be indicated in next section there has also been an independent and parallel development in abstract potential theory.

4. Partial orders and mixed envelopes

In potential theory there are two parallel worlds, the world of mass distributions (measures) and the world of potentials (superharmonic functions with the right behaviour at infinity). It is a matter of taste in which world one prefers to work, because one can always go between them via the bijective maps $\mu \mapsto U^\mu = U * \mu$ and $U^\mu \mapsto \mu = -\Delta U^\mu$.

However, each world has its own partial order $\le$ and these are not identical. Rather, one of them is stronger than the other:

$$\mu_1 \le \mu_2 \text{ implies } U^{\mu_1} \le U^{\mu_2}$$

by the maximum principle. (In two dimensions one has to assume that $\int d\mu_1 = \int d\mu_2$.) In this section we shall work in the world of measures and we shall transfer both partial orders to that side by defining

$$\mu_1 \preccurlyeq \mu_2 \text{ to mean } U^{\mu_1} \le U^{\mu_2} \quad (\text{in all } \mathbb{R}^n).$$

Thus $\preccurlyeq$ is weaker than $\le$:

$$\mu_1 \le \mu_2 \text{ implies } \mu_1 \preccurlyeq \mu_2.$$

One may notice that the two partial orders (or rather the corresponding positive cones) are **polar** to each other with respect to the energy inner product. This means that (allowing signed measures)

$$\mu \succcurlyeq 0 \text{ if and only if } (\mu, \nu)_e \ge 0 \text{ for all } \nu \ge 0,$$

$$\nu \ge 0 \text{ if and only if } (\mu, \nu)_e \ge 0 \text{ for all } \mu \succcurlyeq 0.$$

These statements are immediate from $(\mu, \nu)_e = \int U^\mu d\nu$.

Using the new partial order the definitions and simple properties of partial balayage look even more natural than before. For example, Definition 3.2

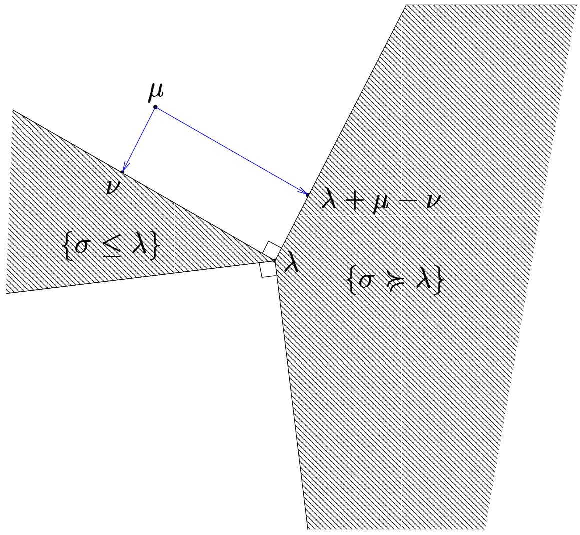

Figure 4: Orthogonal decomposition with respect to a dual pair of cones.

says that $\mathrm{Bal}\,(\mu, \lambda) = \nu$, where ν is the largest, with respect to $\preccurlyeq$, of all measures satisfying

$$\nu \preccurlyeq \mu, \quad \nu \leq \lambda.$$

Such an object is called a **mixed envelope** in a terminology introduced by M. Arsove and H. Leutwiler [7]. See also [28], for example. Following the notation of [7] we can write

$$\mathrm{Bal}\,(\mu, \lambda) = \lambda \curlywedge \mu = \max_{\preccurlyeq}\{\nu : \nu \leq \lambda, \quad \nu \preccurlyeq \mu\}.$$

The characterization (3.6), (3.7), (3.8) of $\mathrm{Bal}\,(\mu, \lambda)$ now takes the form

$$\begin{cases} \nu \preccurlyeq \mu, \\ \nu \leq \lambda, \\ (\mu - \nu, \lambda - \nu)_e = 0, \end{cases}$$

and this can be conceived as an instance of a **Moreau decomposition** [80], i.e., an orthogonal decomposition with respect to a dual pair of convex cones. In our case, $\mu - \lambda$ is decomposed with respect to the cones $\{\sigma : \sigma \succcurlyeq 0\}$ and $\{\sigma : \sigma \leq 0\}$:

$$\mu - \lambda = (\mu - \nu) + (\nu - \lambda).$$

Here (3.6), (3.7), (3.8) say that the two terms are in the respective cones and that the decomposition is orthogonal. This point of view of partial balayage is emphasized in [98]. One may also move the vertex of the cones to λ and say that μ is expressed in terms of its orthogonal projections onto $\{\sigma : \sigma \succcurlyeq \lambda\}$ and $\{\sigma : \sigma \leq \lambda\}$. See Figure 4.

It is clear from the above that there must be also a dual minimization problem for finding ν. Indeed, instead of minimizing $\|\mu-\nu\|_e^2$ among ν with $\nu \le \lambda$, as in Definition 3.1, one may minimize $\|\nu-\lambda\|_e^2$ among ν with $\nu \preccurlyeq \mu$.

As is immediate from the definition, the mixed envelope (or partial balayage) has the monotonicity property with respect to $\preccurlyeq$ that

$$\mu_1 \preccurlyeq \mu_2 \text{ implies } \operatorname{Bal}(\mu_1,\lambda) \preccurlyeq \operatorname{Bal}(\mu_2,\lambda).$$

In addition, one easily infers from (3.12) that

$$\mu_1 \le \mu_2 \text{ implies } \operatorname{Bal}(\mu_1,\lambda) \le \operatorname{Bal}(\mu_2,\lambda).$$

Finally, it is clear that there is a translational invariance:

$$\operatorname{Bal}(\mu+\sigma,\lambda+\sigma) = \operatorname{Bal}(\mu,\lambda)+\sigma,$$

for say $\sigma \in L^\infty$ with compact support. Therefore there is no difficulty in extending the definition of partial balayage to suitable classes of signed measures.

5. Geometry of partial balayage

In this section we shall concentrate on the case $\lambda = m$ and we shall also, from now on, make a slight change of notation: measures will be denoted as distributions, which for example means that Lebesgue measure becomes the function 1 (identically one). If $D \subset \mathbb{R}^n$ is a set, χ_D denotes its characteristic function and the Newtonian potential of χ_D (i.e., of $\chi_D m$) will be denoted U^D. Lebesgue measure (volume) of D will sometimes be denoted $|D|$.

We first discuss the question of when $\operatorname{Bal}(\mu,1)$ takes the pure form

$$\operatorname{Bal}(\mu,1) = \chi_D \tag{5.1}$$

for some open set D, i.e., when there is no remainder term in (3.10).

It is obvious from the definition of $\Omega(\mu)$ that if (5.1) holds, then necessarily $D = \Omega(\mu)$ up to nullsets. More precisely, we get

$$[D] = \Omega(\mu),$$

where $[D]$ denotes the **saturation** of D with respect to Lebesgue measure:

$$[D] = \{x \in \mathbb{R}^n : m(B(x,r)\setminus D) = 0 \text{ for some } r>0\}$$
$$= \mathbb{R}^n \setminus \operatorname{supp}(1-\chi_D).$$

We notice that $D \subset [D] \subset \operatorname{int}\overline{D}$, $|[D]\setminus D| = 0$. If $|\partial D| = 0$, then $[D] = \operatorname{int}\overline{D}$.

The following little lemma, which will be needed in Section 6, gives a direct characterization of (5.1) in terms of potentials.

Lemma 5.1. *Equation* (5.1) *holds if and only if*

$$U^D \le U^\mu \text{ in } \mathbb{R}^n, \tag{5.2}$$

$$U^D = U^\mu \text{ outside } [D]. \tag{5.3}$$

Proof. By (3.6), (3.7), (3.8) with $\nu = \chi_D$, (5.1) is equivalent to

$$U^D \le U^\mu,$$
$$\chi_D \le 1,$$
$$\int (U^\mu - U^D)(1 - \chi_D)dm = 0.$$

Here the second inequality contains no information at all and the third equation is the same as (5.3). Thus the lemma follows. □

In particular it follows that D is uniquely determined (up to nullsets) by μ if (5.2), (5.3) hold. It is not true that D is uniquely determined by (5.3) alone. As an example one may take a uniform measure μ on the unit circle S^1 in $\mathbb{R}^2$. For suitable choices of total mass of μ there will be both a disc and an annulus satisfying (5.3) (only the annulus will satisfy (5.2) then). See Example 1.2 in [95].

If merely (5.3) holds one can still say something in general. In the just mentioned example we have $\partial(\text{disc}) \subset$ annulus, and this is the typical situation, see [95], Theorem 4.7 with corollaries, and also [41] (Cor. 3.1), [46] (Prop. 2.4). Following [46] we state in this direction

Lemma 5.2. *Let $\Omega = \Omega(\mu)$ and let D be any saturated ($D = [D]$) open set satisfying* (5.3). *Then*

$$\partial D \subset \overline{\Omega}. \tag{5.4}$$

Morever $U^\nu \le U^D$, where $\nu = \mathrm{Bal}\,(\mu, 1)$.

Proof. Set $v = U^\nu - U^D$. Since $-\Delta U^\nu \le 1$, v is subharmonic in D and by (5.3), (3.7), $v = U^\nu - U^\mu \le 0$ outside D. It follows that $v \le 0$ everywhere, proving the last statement of the lemma.

Assume next that there exists $x \in \partial D \setminus \overline{\Omega}$ and choose a ball $B = B(x, r)$, $r > 0$, such that $B \cap \overline{\Omega} = \varnothing$. Then $U^\nu = U^\mu$ in B, hence $v = 0$ in $B \setminus D$. From this (and using also the regularity of v derived from (3.12)) follows $\Delta v = 0$ a.e. in $B \setminus D$, whereas $\Delta v = 1 - \nu \ge 0$ in $B \cap D$. Thus v is subharmonic in B.

Using $v \le 0$ we conclude that $0 = v(x) \le \frac{1}{|B|}\int_B v dm \le 0$, hence that $v = 0$ in B. On the other hand $\Delta v = \chi_D - \nu$, which is not identically zero in $B \cap D \ne \varnothing$ since, by definition of $\Omega = \Omega(\mu)$, $\nu = 1$ in no nonempty open subset of $B \subset \Omega^c$.

This is a contradiction and the lemma is proved. □

If (5.3) holds, then μ, D and $\mathrm{Bal}\,(\mu, 1)$ all have the same total mass. It follows that $|D| \ge |\Omega(\mu)|$ (equality if there is no remainder term in (3.10)). Let us call a bounded domain Ω **solid** if $\Omega = \operatorname{int} \overline{\Omega}$ and $(\overline{\Omega})^c$ has only one component (the unbounded one). If $\Omega(\mu)$ is solid then (5.4) together with the comparison between the volumes implies that $D \subset \Omega(\mu)$, $|\Omega(\mu) \setminus D| = 0$.

Thus, if $\Omega(\mu)$ happens to be solid there is, up to nullsets, only one open set D satisfying (5.3). This statement is quite useful for certain uniqueness questions in potential theory. The simplest example is when $\mu = \delta_a$, a point mass. Then $\Omega(\mu)$ is a ball, which is solid. Hence this ball is the unique body with exterior potential equal to U^{δ_a}. More generally, $\Omega(\mu)$ is solid if μ has support in a hyperplane or if μ for example satisfies (*iii*) in Theorem 5.3 below. For further results in this direction, see [95], [41], [46]. For a general orientation on inverse problems in potential theory, see for example [118], [62].

Next we give some sufficient conditions, in terms of μ alone, for (5.1) to hold. Basically what is required is that μ is big enough on its support.

Theorem 5.3. *Equation* (5.1) *holds if μ satisfies any one of the following conditions.*

(i) μ is singular with respect to Lebesgue measure.

(ii) There exists an open set ω such that $\mu \geq 1$ in ω, $\mu = 0$ outside ω.

(iii) μ has support in a ball and the total mass of μ is at least 2^n times the volume of the ball.

Proof. As for the proof we simply give references:

(*i*) is proved in [95] (see the end of section 3 there) and in [41], Theorem 2.4.

(*ii*) is the basic assumption in Theorem 3.7 of [95] and is proved also in [96], [41] (Theorem 2.4). The statement in fact follows from (3.11) (whenever this holds) because $\omega \subset \Omega(\mu)$ by definition of $\Omega(\mu)$ and (3.12).

(*iii*) See Theorem 2 in [101]. The statement can be easily understood from its extremal (worst) case, which is when μ is a point mass on the boundary of the ball. Then $\Omega(\mu)$ will also be a ball, and it has to have the double radius in order to cover the original ball. □

A particular case of (*i*) in Theorem 5.3 is when μ is a finite sum of point masses. In the case of two dimensions the boundary of $\Omega = \Omega(\mu)$ will then be an algebraic curve, see [4], [37]. As an example, using complex variable notations ($z = x + iy$ etc.) and taking

$$\mu = \pi r^2(\delta_{-1} + \delta_{+1}) \tag{5.5}$$

for some $r > 1$, gives the equation

$$(x^2 + y^2)^2 - 2r^2(x^2 + y^2) - 2(x^2 - y^2) = 0$$

for $\partial\Omega$. See Figure 5, which also illustrates Theorem 5.4 below. This Ω can be viewed as the two discs $B(-1, r)$, $B(1, r)$ potential theoretically glued together. In fact, as an instance of (3.13), the same Ω is gotten from

$$\mu = \chi_{B(-1,r)} + \chi_{B(1,r)}.$$

Recall that $\mathrm{Bal}\,(\pi r^2\delta_a, 1) = \chi_{B(a,r)}$.

For the above two point measure μ the polynomial for $\partial\Omega$ could be written down explicitly, but in general, with $\mu = \sum_{j=1}^{m} a_j\delta_{z_j}$ and $m \geq 3$, no effective method is known for finding the polynomial for $\partial\Omega$ from the data of μ. Special cases with many symmetries can however sometimes be handled, see

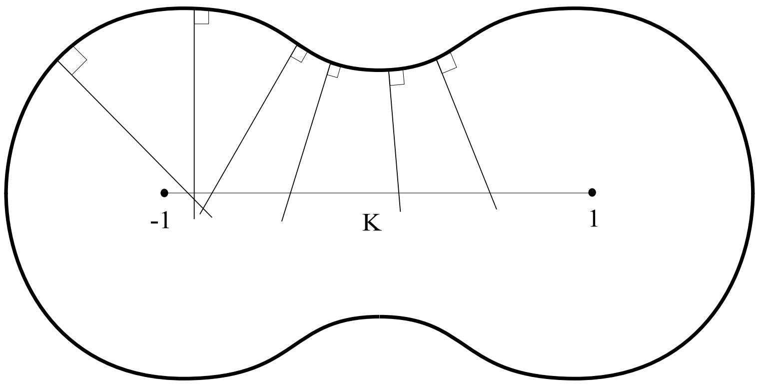

Figure 5: Partial balayage of two point masses.

for example [19], [20]. In higher dimensions very little is known whatsoever about algebraicity of $\partial\Omega$. See however [64], [114].

The above mentioned results about algebraic boundaries in two dimensions hold more generally for domains $D \subset \mathbb{C}$ satisfying the weaker form of graviequivalence

$$\nabla U^D = \nabla U^\mu \text{ outside } D, \tag{5.6}$$

and then even with μ allowed to be any complex-valued distribution with support in a finite number of points. Apart from a complex conjugation and a constant factor the field ∇U^μ is in two dimensions the same thing as the **Cauchy transform** of μ:

$$\hat{\mu}(z) = \frac{1}{\pi}\int \frac{d\mu(\zeta)}{z-\zeta} = -4\frac{\partial U^\mu}{\partial z}.$$

Here $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. That μ is a distribution with support in a finite number of points means exactly that $\hat{\mu}$ is a rational function, and (5.6) therefore expresses in this case that the Cauchy transform of D (i.e., of χ_D) agrees with a rational function outside D.

Another way to express (5.6) is to say that

$$\int_D \varphi dm = \langle \mu, \varphi \rangle$$

holds for every integrable analytic function φ in D, and D is then called a **quadrature domain** for analytic functions [4], [95], [107]. The bracket $\langle \cdot, \cdot \rangle$ denotes the action of a distribution on a test function. There are similar notions of quadrature domains for harmonic and subharmonic functions, corresponding to the graviequivalence statements (5.3) and (5.1) (i.e., (5.2) plus (5.3)) respectively.

Returning to the general case that μ is a measure in $\mathbb{R}^n$, set

$$K = \operatorname{conv}\operatorname{supp}\mu,$$

the convex hull of the (closed) support of μ. From what was said in connection with Theorem 5.3 it is not surprising that $\operatorname{Bal}(\mu, 1)$ will always be of the form χ_D outside K, i.e., that the remainder term in (3.10), if there is one, must be located inside K (in case $\lambda = m$). This is part of

Theorem 5.4. *Let* $\nu = \operatorname{Bal}(\mu, 1)$, $\Omega = \Omega(\mu)$. *Then*

(i)

$$\operatorname{Bal}(\mu, 1)|_{\mathbb{R}^n \setminus K} = \chi_{\Omega \setminus K}.$$

(ii) $\partial\Omega \setminus K$ *is smooth real analytic.*

(iii) Everywhere in $\Omega \setminus K$ *the gradient* $\nabla(U^\nu - U^\mu)$ *is nonzero and directed towards* K.

(iv) For any $x \in \partial\Omega \setminus K$ *the inward normal ray* N_x *of* $\partial\Omega$ *at* x *intersects* K.

In dimension $n = 2$ *we have, in addition,*

(v) The normal rays N_x *in (iv) do not intersect each other before they reach* K.

(vi) There exist radii $r = r(x) > 0$ *for* $x \in K \cap \Omega$ *such that*

$$\Omega = \cup_{x \in K \cap \Omega} B(x, r(x)).$$

One naturally conjectures that (v) and (vi) hold also in higher dimensions, but there is no proof at present. In two dimensions we know of two proofs, [48] and [49]. Statements (v) and (vi) are actually equivalent and they give a natural upper bound on the curvature of $\partial\Omega \setminus K$.

As to (ii), $\partial\Omega$ may very well have (analytic) singularities inside K. For example, taking $r = 1$ in (5.5) gives a pair of touching discs, and the touching point is a singular point of $\partial\Omega$ belonging to K.

In two dimensions one knows exactly what type of singularities $\partial\Omega$ may have in $K \setminus \operatorname{supp}\mu$ see [99], [100]. In [45] a global real analytic defining function of $\partial\Omega \setminus \operatorname{supp}\mu$ was produced using an exponential transform. Also in higher dimension much is known, see [12], [13], [15].

Proof. Let us just outline the proof of (i)-(iv), which is based on a variant of the "moving plane method" [105], [35]. Set

$$u = U^\mu - U^\nu.$$

Then, by Definition 2, u is the smallest of all functions satisfying

$$u \geq 0, \quad \Delta u \leq 1 - \mu.$$

Moreover, by (3.9),

$$u = 0 \text{ on } \Omega^c,$$

and since u can be shown to be continuously differentiable outside $\operatorname{supp}\mu$ (this follows from (3.12)) and zero is the minimum value of u it follows that

$$\nabla u = 0 \text{ on } \Omega^c \setminus \operatorname{supp}\mu.$$

In particular this holds on $\partial\Omega \setminus \operatorname{supp}\mu$.

The theorem only contains assertions of what happens outside K, i.e., about what happens at points which can be separated from $\operatorname{supp}\mu$ by a hyperplane. Assume that this hyperplane is $\{x \in \mathbb{R}^n : x_n = 0\}$, that $\operatorname{supp}\mu$ is contained in $\{x_n < 0\}$, and then we shall say something about

$$\Omega^+ = \Omega \cap \{x_n > 0\}.$$

Let u^* be the reflection of u in $\{x_n = 0\}$, i.e., $u^*(x', x_n) = u(x', -x_n)$, and set

$$v = u - \inf(u, u^*) = (u - u^*)_+.$$

Since $\Delta u \leq 1 - \mu \leq 1$ we have $\Delta u^* \leq 1$. Therefore $\Delta \inf(u, u^*) \leq 1$ everywhere. Since $\Delta u = 1$ in Ω^+ it follows that $\Delta v \geq 0$ in Ω^+. Moreover $v = 0$ on $\partial(\Omega^+)$.

Thus we conclude (maximum principle) that $v \leq 0$ in Ω^+, Thus u is smaller (or at least not larger) at any point above the hyperplane $\{x_n = 0\}$ than at the reflected point below it. On the hyperplane this gives

$$\frac{\partial u}{\partial x_n} \leq 0 \text{ on } \{x_n = 0\}.$$

Since $\nabla u = 0$ on $(\partial\Omega)^+$ and $\Delta \frac{\partial u}{\partial x_n} = \frac{\partial}{\partial x_n}\Delta u = 0$ in $\Omega \cap \{x_n > 0\}$ we can apply the maximum principle again, now to $\partial u / \partial x_n$, to obtain

$$\frac{\partial u}{\partial x_n} \leq 0 \text{ in } \Omega^+.$$

An easy argument shows that the inequality is everywhere strict.

From the above everything follows: u is strictly decreasing as a function of x_n in Ω^+ and on $\partial\Omega$ u vanishes. Thus $(\partial\Omega)^+$ is a graph of a function when seen from $\{x_n = 0\}$. To this graph general regularity results for free boundaries [12], [13], [30], [15] apply, with the conclusion that it is real analytic. Also, the vector $-\nabla u = \nabla(U^\nu - U^\mu)$ and the ray N_x for $x \in (\partial\Omega)^+$ are both directed towards $\{x_n = 0\}$.

Applying the above arguments to all hyperplanes separating parts of $\Omega \setminus K$ from K easily gives the statements (i)-(iv) of the theorem.

□

6. Continuous balayage and Hele-Shaw flow

By introducing a time parameter classical balayage can be used as the infinitesimal generator of a process which turns out to be an instance of partial balayage performed continuously in time. In physics the same sort of process appears in Hele-Shaw flow with a free boundary, various electrochemical processes, problems of melting/freezing under simplified assumptions etc.

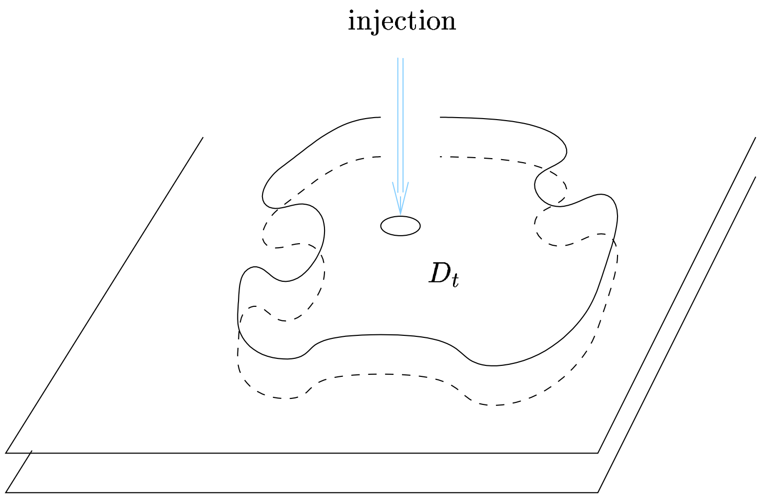

Figure 6: Hele-Shaw flow with injection.

Let us describe the standard version of the Hele-Shaw problem in this context [33], [87], [26], [59], [115]. A Hele-Shaw flow [55], [56] is the flow of a viscous incompressible fluid (e.g. oil) in the narrow gap between two parallel plates. In the two-dimensional view, taking averages across the gap, this turns out to be a potential flow with the fluid pressure as the potential function.

In our case the fluid should only occupy a finite region at each instant of time and the fluid boundary (in the two-dimensional view) should be free, which means that the pressure there should be constant (say zero). The driving force will be suitable sources, e.g. that more fluid is injected through one or several holes in one of the plates. See Figure 6.

In the simplest case, injection at one point call it a, the mathematical model becomes the following. An initial (bounded) domain $D_0 \subset \mathbb{R}^2$ is given with $a \in D_0$. If D_t denotes the region of fluid at time t, the fluid pressure in D_t will be (up to normalization) the Green function $G_{D_t}(\cdot, a)$ of D_t with pole at a. It follows that the fluid velocity will be proportional (let us say equal) to the gradient $-\nabla G_{D_t}(\cdot, a)$ and hence that ∂D_t has to move with velocity $-\nabla G_{D_t}(\cdot, a)|_{\partial D_t}$. (Some smoothness of ∂D_t is needed for this to make sense.) In other words, ∂D_t is to move with normal velocity $-\partial G_{D_t}(\cdot, a)/\partial n$.

Now we recall (2.6) that $-\partial G_{D_t}(\cdot, a)/\partial n$ is the same thing as the density of harmonic measure ω_a of ∂D_t with respect to a, which in turn is an instance of classical balayage: $\omega_a = \mathrm{Bal}\,(\delta_a, D_t^c)$. On the other hand, as a general fact, the normal velocity of a propagating boundary ∂D_t equals the density with respect to arclength measure on ∂D_t of the distributional derivative $\frac{d}{dt}\chi_{D_t}$

(just consider the differential quotients to realize this). Thus we arrive at the **Hele-Shaw law** for the motion of ∂D_t in a distributional form:

$$\frac{d}{dt}\chi_{D_t} = \operatorname{Bal}(\delta_a, D_t^c). \tag{6.1}$$

One may call this "motion by harmonic measure". Clearly (6.1) makes sense in any number of dimensions, and we shall henceforth have no restriction on the dimension.

The forward Hele-Shaw problem is that of finding the evolution $\{D_t : 0 \leq t < \infty\}$ (or at least $0 \leq t < \varepsilon$) governed by (6.1) when D_0 is given. If all the ∂D_t are smooth and depend smoothly on t there is no difficulty in giving a precise pointwise meaning to (6.1), and we call such an evolution a **strong** (or **classical**) **solution** of the Hele-Shaw problem. Clearly any strong solution will be monotone increasing:

$$D_s \subset D_t \text{ for } s < t.$$

Theorem 6.1. *Assume $\{D_t : 0 \leq t < \infty\}$ is a strong solution of the Hele-Shaw problem. Then*

$$\operatorname{Bal}(t\delta_a + \chi_{D_0}, 1) = \chi_{D_t}. \tag{6.2}$$

Proof. Formally one may just integrate (6.1) with respect to time from $t = 0$ to an arbitrary $t > 0$ to obtain

$$\chi_{D_t} - \chi_{D_0} = \operatorname{Bal}(t\delta_a, \chi_{D_0^c}),$$

which becomes (6.2) after adding χ_{D_0} to both members. However, the appearence of the density $\chi_{D_0^c} = 1 - \chi_{D_0}$ above is not that easy to motivate, so let us indicate also a more genuine derivation of (6.2).

A useful way to express (6.1) is to say that for any test function φ

$$\frac{d}{dt}\int_{D_t} \varphi dm = \int_{\partial D_t} \varphi(-\frac{\partial G_{D_t}(\cdot, a)}{\partial n})ds. \tag{6.3}$$

Choosing φ to be harmonic in D_t this becomes

$$\frac{d}{dt}\int_{D_t} \varphi dm = \varphi(a), \tag{6.4}$$

and upon integrating from $t = 0$ to an arbitrary $t > 0$,

$$\int_{D_t} \varphi dm - \int_{D_0} \varphi dm = t\varphi(a). \tag{6.5}$$

Now, a good harmonic function in D_t (and all D_s with $0 \leq s < t$) is the Newton kernel with pole outside D_t, namely $\varphi(x) = U(x - y)$ with $y \notin D_t$. Then (6.5) becomes

$$U^{D_t}(y) - U^{D_0}(y) = U^{t\delta_a}(y).$$

Thus

$$U^{D_t} = U^{D_0 + t\delta_a} \text{ outside } D_t. \tag{6.6}$$

Similarly, taking φ to be just superharmonic in D_t one gets (6.4) and (6.5) with the equality replaced by $\leq$. Then it is allowed to take $\varphi(x) = U(x-y)$ for any $y \in \mathbb{R}^n$, which gives

$$U^{D_t} \leq U^{D_0+t\delta_a} \text{ in } \mathbb{R}^n. \tag{6.7}$$

But now we are done because Lemma 5.1 tells that (6.2) is equivalent to (6.6), (6.7) taken together.

□

We may call any family $\{D_t : 0 \leq t < \infty\}$ of domains (or open sets) satisfying (6.2) a **weak solution** of the Hele-Shaw problem. There are several advantages with this concept of weak solution. First, it *always exists*, for any (bounded) initial domain D_0. Indeed, (ii) of Theorem 5.3 shows that $\operatorname{Bal}(t\delta_a + \chi_{D_0}, 1)$ always is of the form χ_{D_t} for some D_t. Second, it is *unique* if one disregards differences of nullsets (or one may normalize D_t to be saturated: $D_t = [D_t]$). Third, it is *global in time* ($0 \leq t < \infty$). Finally, any D_t, $t > 0$, can be obtained *directly* from D_0 (the intermediate domains need not be computed).

As a corollary of Theorem 6.1 we see that on any interval $0 \leq t < \varepsilon$ there exists at most one strong solution of the Hele-Shaw problem. Existence of strong solutions is a more delicate question. A local, in a short two-sided time interval $-\varepsilon < t < \varepsilon$, strong solution exists if and only if ∂D_0 is smooth real analytic [116], [86], [29], [111], [112].

Some further good properties of weak (and *a fortiori* strong) solutions of the Hele-Shaw problem is that they regularize in time. Asymptotically, as $t \to \infty$, they approach circular shape, although in the meantime they may very well undergo topological changes. All this follows from Theorem 5.4: denoting by K the closed convex hull of $\overline{D_0}$ we have that $\partial D_t \setminus K$ is always smooth analytic, the inward normals from points on $\partial D_t \setminus K$ intersect K and (in the case of two dimensions) do not intersect each other before that. If the initial domain D_0 is starlike with respect to the injection point, then D_t remains starlike for all $t > 0$ [22] (Theorem 4.1), [50] (Theorem 3.12), [60]. See also [75]. Thus no topological changes can occur in this case, and one can show that the weak solution is actually a strong solution on all $0 < t < \infty$ [44], [51].

The easiest way to understand the good regularizing properties of the forward Hele-Shaw problem is perhaps via its probabilistic formulation. Recall that harmonic measure $\omega_a = \operatorname{Bal}(\delta_a, D^c)$, which is a (positive) measure of total mass one on ∂D, gives the probability distribution for the first place of exit from D for a Brownian motion particle started at the point $a \in D$ (if $E \subset \partial D$, the probability that the particle will reach ∂D for the first time at some point of E is $\omega_a(E)$).

Therefore one can think of the Hele-Shaw problem in probabilistic terms and, for simplicity, in discrete time and space as follows: at each unit of time a Brownian motion particle is emitted at $a \in D$. This moves around until it reaches ∂D, and there it "eats up" a unit piece (in terms of volume) of D^c and then dies. Then comes next random particle, which eats up another piece of what remains of D^c, etc..

From this it is clear that ∂D_t will become smoother and smoother and approach circular shape, because if some piece of ∂D_t is highly exposed (like ∂D_t having an inward cusp) or is very close to a, then this piece will have a higher probability of being eaten up quickly. It is equally clear that driving the Hele-Shaw problem backwards in time will result in an ill-posed and unstable process. This will be discussed a little more in Section 7.

An extensive bibliography on Hele-Shaw type problems can be found at [36]. For the Hele-Shaw problem related to potential theory, see also [78].

For the rest of this section we shall mention some variants of the Hele-Shaw problem. First, Hele-Shaw flow with a more general source term than δ_a has a corresponding description in terms of partial balayage. With the sources represented by a measure μ, (6.2) becomes

$$\mathrm{Bal}\,(t\mu + \chi_{D_0}, 1) = \chi_{D_t} \tag{6.8}$$

($t > 0$). It is convenient, but not absolutely necessary, to assume that the sources are located in D_0 because then *(ii)* of Theorem 5.3 ensures that $\mathrm{Bal}\,(t\mu + \chi_{D_0}, 1)$ really is of the form χ_{D_t} for some open set D_t. In the case $\mu = \delta_a$ we may however very well let the point a be outside D_0 (for weak solutions). Still more general forms of continuous balayage are obtained by replacing $t\mu$, or even $t\mu + \chi_{D_0}$, by a more general increasing family of measures $\mu(t)$.

For continuous balayage the first of the two conditions in Lemma 5.1 is often automatically satisfied if the family D_t is assumed *a priori* to be monotone or continuous, see [95], Theorem 10.13 and Corollary 10.14. We may state a result in this direction as follows.

Theorem 6.2. *Let $\mu(t)$, $t \geq 0$, be a monotone increasing family of measures (i.e., $\mu(s) \leq \mu(t)$ for $s < t$) with the total masses $\int d\mu(t)$ continuously increasing in t. Let $\{D_t : t \geq 0\}$ be a family of open sets such that*

$$\mathrm{Bal}\,(\mu(0)) = \chi_{D_0},$$

$$U^{D_t} = U^{\mu(t)} \text{ outside } D_t.$$

Then, under some mild regularity assumption on e.g. $\mu(0)$ (see the proof), we have

$$\mathrm{Bal}\,(\mu(t)) = \chi_{D_t} \quad (t > 0)$$

provided any one of the following additional conditions hold:

(i) D_t is continuous in t in the sense that $t \mapsto |D_t \cap B|$ is continuous for every ball B.

(ii) $\{D_t\}$ is monotone:

$$D_s \subset D_t \text{ for } s < t.$$

Proof. It is enough to prove that $D_t \subset \Omega_t$, where $\Omega_t = \Omega(\mu(t))$ is the saturated set for $\mathrm{Bal}(\mu(t))$, because then necessarily $D_t = \Omega_t$ a.e. and everything follows. By Lemma 5.2

$$\partial D_t \subset \overline{\Omega_t}. \tag{6.9}$$

From this we first conclude

$$D_t \subset \overline{\Omega_t} \quad (t \geq 0), \tag{6.10}$$

as follows.

By assumption, (6.10) holds for $t = 0$. If (6.10) fails for some $t > 0$, pick a small ball $B \subset D_t \setminus \overline{\Omega_t}$. Since $\{\Omega_t\}$ is monotone increasing it follows that $B \cap \overline{\Omega_s} = \varnothing$ for all $0 \leq s \leq t$. For $s = 0$ this gives $B \cap D_0 = \varnothing$. But $B \cap D_t = B$ and by (6.9) we have $B \cap \partial D_s \subset B \cap \overline{\Omega_s} = \varnothing$ for all $s \leq t$. Thus $s \mapsto |D_s \cap B|$ can not be continuous.

In summary, we have proved (6.10) in case (i) holds. Using that $|D_t| = \int d\mu(t)$ is continuous in t one finds that (ii) implies (i), so (6.10) holds in any case (i) or (ii).

Now (6.10) implies $D_t = \Omega_t$ (a.e.) if just $|\partial\Omega_t| = 0$ (zero volume). It is for this that some mild extra assumption is needed. If e.g. $\mu(t) = t\delta + \chi_{D_0}$ it is more than enough to assume that D_0 has C^1 boundary. And it is always true [14], [66] (Lemma 2.11) that $|\partial\Omega_t \setminus \mathrm{supp}\,(\mu(t))| = \varnothing$. □

For the one point injection at the origin in two dimensions we conclude the following.

Corollary 6.3. *Let $\{D_t : t \geq 0\}$ be simply connected domains with C^1 boundaries in $\mathbb{R}^2 = \mathbb{C}$ such that*

$$\int_{D_t} z^k dm = \int_{D_0} z^k dm \quad (k = 1, 2, \dots),$$

$$|D_t| = |D_0| + t.$$

Here $z = x + iy$. Then

$$\mathrm{Bal}\,(t\delta + \chi_{D_0}) = \chi_{D_t},$$

if $\{D_t\}$ satisfies (i) or (ii) in Theorem 6.2.

The integrals of motion $\int_{D_t} z^k dm$ $(k = 1, 2, \dots)$ for this Hele-Shaw problem were discovered by S. Richardson [87]. The corollary says that the conservation of these complex moments characterizes a weak solution (under the stated assumptions and up to scaling of time). The solution in the corollary will actually be a strong solution for $t > 0$ because the assumptions are enough to guarantee the existence of such a solution.

Proof. The set of equations is equivalent to

$$\int_{D_t} \varphi(z) dm = \int_{D_0} \varphi(z) dm + t\varphi(0)$$

holding for all $\varphi(z) = 1, z, z^2, \dots$, hence for all linear combinations of these functions, hence for sufficiently many analytic and (taking real and imaginary parts) harmonic functions so that any Newton kernel $\varphi(z) = U(z-\zeta)$ with $\zeta \notin D_t$ can be approximated. Thus $U^{D_t} = U^{D_0} + U^{t\delta}$ outside D_t. Now the theorem can be applied. □

Next, one may study Hele-Shaw flow (and partial balayage in general) with the density (weight) $\rho = 1$ replaced by other weights. In analytic function theory (hence $n = 2$) one often runs into weights of the kind $\rho = |f|^2$ where f is an analytic function (which may have zeros). See e.g. [95] (the appendix), [52], [11], [63].

Let us just mention one recent result in this context. Note that the weight $\rho = |f|^2$ (with f analytic) is logarithmically subharmonic: $\Delta \log \rho \geq 0$. In [52], [53] the authors prove that for any logarithmically subharmonic weight ρ satisfying certain smoothness conditions, the solution family $\{D_t : t \geq 0\}$ for the weighted Hele-Shaw problem, starting from empty space,

$$\mathrm{Bal}\,(t\delta_a, \rho) = \rho\chi_{D_t}$$

remains simply connected for all $t > 0$, and is in fact a strong solution.

In [53] this result is applied to exhaust a hyperbolic manifold (in two dimensions) by simply connected domains and providing it with a natural system of polar coordinates. Other topological results for Hele-Shaw flow on Riemannian manifolds (like surfaces embedded in $\mathbb{R}^3$) can be found in [114].

Another variant of Hele-Shaw flow is "motion by equilibrium measure". This means that the fluid region D_t contains a neighbourhood of infinity with a source (or sink) there. In the notation of (2.8) the governing equation will be

$$\frac{d}{dt}\chi_{D_t} = \mathrm{Bal}\,(\alpha\delta_\infty, D_t^c).$$

Integrating this as before should give, in principle,

$$\chi_{D_t} = \mathrm{Bal}\,(\chi_{D_0} + t\alpha\delta_\infty, 1). \tag{6.11}$$

However, the right member here does not make obvious sense because D_0 is too large. Also, δ_∞ has only a symbolic meaning. However everything can be resolved, either by replacing the Newton kernel with a modification of it which has better properties at infinity, see [65], [23], or by working in terms of the complements $K_t = D_t^c$ instead.

In view of Lemma 5.1, (6.11) should be equivalent (formally) to $U^{D_t} \leq U^{D_0 + t\alpha\delta_\infty}$ holding everywhere with equality outside D_t. Working with the compacts K_t instead of D_t, what one really comes up with, and which makes good sense, is the system

$$U^{K_t} \geq U^{K_0} - \gamma_t \text{ in } \mathbb{R}^n, \tag{6.12}$$

$$U^{K_t} = U^{K_0} - \gamma_t \text{ in } K_t. \tag{6.13}$$

Here

$$\gamma_t = \int_0^t \beta_s ds = \alpha \int_0^t \frac{ds}{\operatorname{Cap}(K_s)},$$

where β_s is the equilibrium constant (the potential on K_s) for the mass α on K_s and $\operatorname{Cap}(K_s) = \frac{\alpha}{\beta_s}$ the capacity, see (2.9).

To actually derive (6.12), (6.13) one may use the Hele-Shaw law in a form analogous to (6.3), namely

$$\frac{d}{dt}\int_{K_t} \varphi dm = \int_{\partial K_t} \varphi \frac{\partial p_t}{\partial n} ds,$$

where p_t is the equilibrium potential for the mass α on K_t and φ ranges over suitably regular test functions (so that the equation makes sense). Taking $\varphi(x) = U(x-y)$ with y on either side of ∂K_t gives (6.12), (6.13).

The interpretation (6.12), (6.13) of (6.11) is the weak formulation of the motion by equilibrium measure problem forward in time. Taking the gradient of (6.13) we see that

$$\nabla U^{K_0 \setminus K_t} = 0 \text{ in } K_t,$$

i.e., that the difference region $K_0 \setminus K_t$ is a **null-cavity domain**, cf. [97]. This is of course just the integrated version of the fact that the equilibrium distribution on K_t produces no gravitational (or electrostatic) field inside K_t. See also [103].

It also follows that K_t will disappear in finite time and that the last surviving point(s) will be points where U^{K_0} attains its maximum. Hence $\nabla U^{K_0} = 0$ at these extinction points. See [27], [114].

Let us say just a few words about the ill-posed problem obtained when motion by equilibrium measure is driven backwards in time. It has been shown in [61], [23] that the only cases in which a solution K_t, even in a weak sense, exists for all $-\infty < t < 0$ and exhausts all $\mathbb{R}^n$ as $t \to -\infty$ is when K_0 is an ellipsoid (or a degenerated version of a such). Indeed, K_0 needs to be what is called a **null-quadrature domain**, see [94], [31]. There however exists an abundance of solutions which do not exhaust $\mathbb{R}^n$ but still exist for all $-\infty < t < 0$, see [23].

The probabilistic version of the ill-posed version of motion by equilibrium measure has been studied under the name DLA, diffusion-limited aggregation, at least in two dimensions. Here Brownian motion particles are emitted at infinity, walk around at random and eventually (in case $n = 2$) reach the "aggregate" K_t and attach to it. One may think of K_t as a growing crystal of ice, for example. See [117], [69], [16].

7. Inverse balayage and potential theoretic skeletons

Continuing the discussion of the Hele-Shaw problem (and now back to bounded fluid regions), there is a version of it which is more canonical than the others because there is no particular source term. This is the squeezing version: the dynamics of D_t is produced by squeezing the two plates together so that at time t the distance between them is, say, e^{-t}. Cf. [77], [112].

This turns out to be equivalent to having a uniform source on D_t for each t, and the infinitesimal law, replacing (6.1), will therefore be

$$\frac{d}{dt}\chi_{D_t} = \mathrm{Bal}\,(\chi_{D_t}, D_t^c). \tag{7.1}$$

We leave it as an exercise for the reader (or see [42]) to show that the integrated version of this is

$$e^{-t}\chi_{D_t} = \mathrm{Bal}\,(e^{-s}\chi_{D_s}, e^{-t})$$

for any $s < t$, or equivalently

$$\chi_{D_t} = \mathrm{Bal}\,(e^{t-s}\chi_{D_s}, 1). \tag{7.2}$$

Perhaps (7.2) looks rather more plausible than (7.1), so one may equally well believe in (7.2) directly.

In any case, (7.2) will be the starting point for the discussions in this section. If D_0 is given, then D_t for any $t > 0$ can be produced (uniquely) by taking $s = 0$ in (7.2):

$$\chi_{D_t} = \mathrm{Bal}\,(e^{t}\chi_{D_0}, 1).$$

If the chain $\{D_t\}$ also involves negative values of t we see that these D_t have to satisfy

$$\chi_{D_0} = \mathrm{Bal}\,(e^{-t}\chi_{D_t}, 1)$$

($t < 0$). However, this equation by no means determines D_t uniquely from D_0.

Now assume that D_0 is such that there exists a global, in both time directions, strong solution $\{D_t : -\infty < t < \infty\}$ of (7.1). This is a relatively rare situation, but there are a number of interesting examples. The domains D_t will then be homeomorphic to balls because strong solutions are not allowed to change topology. Consider the mass distributions

$$\mu(t) = e^{-t}\chi_{D_t}.$$

By (7.2) they are all graviequivalent. Indeed, Lemma 5.1 tells us that (7.2) is equivalent to

$$U^{\mu(s)} \leq U^{\mu(t)} \text{ in } \mathbb{R}^n,$$
$$U^{\mu(s)} = U^{\mu(t)} \text{ outside } D_t$$

($s < t$). We also have

$$\mu(t) \geq 0,$$
$$\operatorname{supp}\mu(t) = \overline{D_t},$$
$$|D_t| = e^{t}|D_0| \to 0 \text{ as } t \to -\infty.$$

It follows that, as $t \to -\infty$, the measures $\mu(t)$ converge weakly to some limit measure μ satisfying

$$\mu \geq 0,$$
$$|\operatorname{supp}\mu| = 0.$$

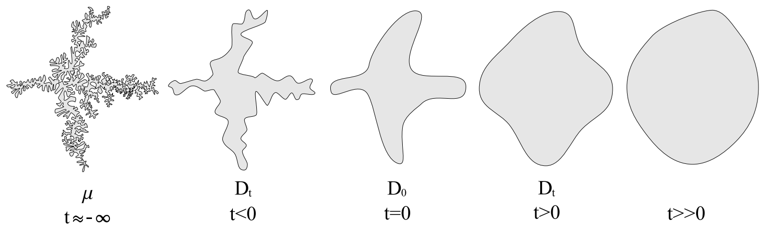

Figure 7: Squeezing version of Hele-Shaw problem, from skeleton to a round blob.

(We are stating positivity of μ explicitly because it will be useful at times to be able to relax on this condition.) We also get, from (7.2),

$$\chi_{D_t} = \mathrm{Bal}\,(e^t\mu, 1) \quad (-\infty < t < \infty), \tag{7.3}$$

in particular

$$\chi_{D_0} = \mathrm{Bal}\,(\mu, 1). \tag{7.4}$$

In addition to the above, since all the D_t had to be topological balls, the topology of $\mathrm{supp}\,\mu$ will be such that it does not disconnect $\mathbb{R}^n$.

It is natural to think of the μ obtained above as a potential theoretic skeleton for D_0, and this is the point of view we shall take from now on. We then relax on the smoothness assumptions to allow for weak solutions of the squeezing Hele-Shaw problem. Given any bounded domain $\Omega \subset \mathbb{R}^n$ (which is considered as a body of density one) we say that μ is a **potential theoretic skeleton** or **mother body** for Ω if it satisfies the following conditions ("axioms").

$(M1)$ $\quad U^\Omega = U^\mu$ outside Ω,

$(M2)$ $\quad U^\Omega \le U^\mu$ in $\mathbb{R}^n$,

$(M3)$ $\quad \mu \ge 0$,

$(M4)$ $\quad |\mathrm{supp}\,\mu| = 0$,

$(M5)$ $\quad \mathrm{supp}\,\mu$ does not disconnect any part of Ω from $(\overline{\Omega})^c$.

The last axiom says more exactly that for any $x \in \Omega \setminus \mathrm{supp}\,\mu$ there is an arc in $\mathbb{R}^n \setminus \mathrm{supp}\,\mu$ connecting x to some point in $\mathbb{R}^n \setminus \overline{\Omega}$.

Conditions $(M1)$, $(M4)$ and $(M5)$ are quite demanding and express that the potential of Ω has a harmonic continuation, given by U^μ, from $(\overline{\Omega})^c$ to all of $\mathbb{R}^n$ minus a closed nullset, namely $\mathrm{supp}\,\mu$. Condition $(M2)$ complements $(M1)$ so that they together say that

$$\chi_\Omega = \mathrm{Bal}\,(\mu, 1). \tag{7.5}$$

The term "mother body" (or "maternal body") comes from the Bulgarian school of geophysics led by D. Zidarov [119] and probably is thought of as indicating that it is something which generates the original body Ω, as in (7.5). However, the pictures we have of mother bodies rather remind of skeletons than of healthy mothers.

If Ω has a mother body μ it is natural to consider the D_t given (up to nullsets) by

$$\chi_{D_t} = \mathrm{Bal}\,(e^t\mu, 1)$$

($-\infty < t < \infty$) as a weak solution of the squeezing Hele-Shaw problem with initial domain $D_0 = \Omega$. However, it is relatively rare that a mother body exists, and when it exists it need not be unique. Indeed, the problem of finding a mother body, as well as any Hele-Shaw problem backward in time, is severely ill-posed and unstable. Nevertheless, there do exist interesting and general classes of domains for which mother bodies exist, classes which are dense among all domains in terms of ordinary topologies (e.g. Hausdorff distance between domains). These mother bodies are on the other hand extremely sensitive to small perturbations of the domains (see Example 5 below).

One way to think of (7.5), even if μ does not satisfy all of the axioms for a mother body, is that μ stores the information of Ω in a compact form. This is useful not only for the backward Hele-Shaw problem but generally if one wishes to make smooth variations of Ω. If for example μ is of the form $\mu = (1+\rho)\chi_\omega$ for some subdomain $\omega \subset \Omega$ and some positive function ρ in ω, then one can by (ii) of Theorem 5.3 vary ρ freely among such functions and all the time obtain a domain $\Omega = \Omega_\rho$ as in (7.5). When $\partial\Omega$ is smooth real analytic such a measure μ, with $\overline{\omega} \subset \Omega$, can always be found (see [39]) and the smoothness of the domain variation follows from [104], [62] (Corollary 5.1.4) (see also [14]). Domain variations of this kind have been used in [95] (section 4), [40], [41] (Theorem 4.3) and they can in particular be used to prove local existence of strong solutions of Hele-Shaw problems.

We now give some examples of mother bodies.

Example 1. For $\Omega = B$, the unit ball in $\mathbb{R}^n$, $\mu = |B|\delta$ is the unique mother body.

Example 2. Let Ω be the ellipse in $\mathbb{R}^2$ given in terms of the coordinates x and y by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1,$$

where $a > b > 0$. Then the measure μ on the focal segment $[-c, c]$ ($c = \sqrt{a^2 - b^2}$) defined by

$$d\mu = \frac{2ab}{c^2}\sqrt{c^2 - x^2}dx \qquad (-c < x < c)$$

is a mother body for Ω, and it is unique [110]. See Figure 8. The inward normals illustrate Theorem 5.4.

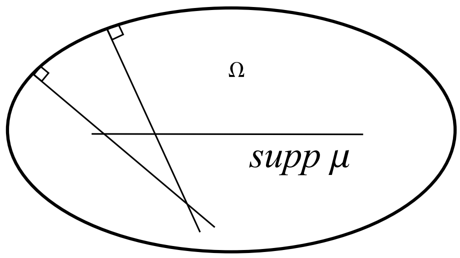

Figure 8: The mother body of an ellipse.

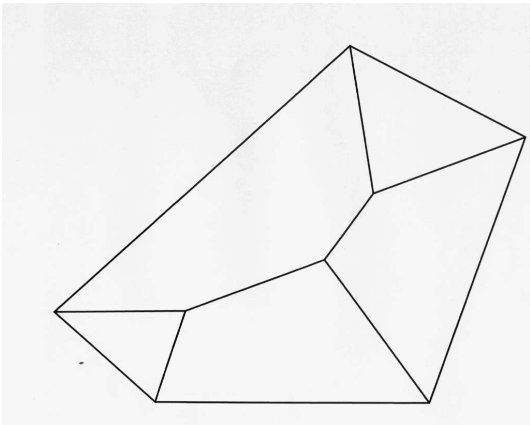

Figure 9: The ridge of a convex polyhedron.

Similarly, for any ellipsoid in higher dimensions there is a mother body sitting on the focal ellipsoid (an ellipsoid of codimension at least one) [70], [71].

EXAMPLE 3. Every convex polyhedron $\Omega \subset \mathbb{R}^n$ has a unique mother body. It sits on the **ridge** of the polyhedron, i.e., the set of those points in Ω which have at least two closest neighbours on $\partial\Omega$. See Figure 9.

It might be instructive to sketch the proof of the just mentioned fact (more details can be found in [43]). Constructing the mother body, μ, amounts to the same thing as constructing the function

$$u = U^{\mu} - U^{\Omega}.$$

This should satisfy, in particular,

$$\begin{aligned} &u \geq 0, \\ &u = 0 \text{ outside } \Omega, \\ &\Delta u = \chi_\Omega - \mu. \end{aligned}$$

Here the first two requirements correspond to conditions $(M2)$ and $(M1)$ in the definition of a mother body while the third condition gives the relationship between u and μ.

Let $d(x, \Omega^c)$ denote the distance from a point $x \in \mathbb{R}^n$ to the complement of Ω. Defining u by

$$u(x) = \frac{1}{2} d(x, \Omega^c)^2$$

the above three properties hold with

$$\mu = \chi_\Omega - \Delta(\frac{1}{2} d(\cdot, \Omega^c)^2).$$

In a neighbourhood of any point in Ω which have only one closest neighbour on $\partial\Omega$, $d(\cdot, \Omega^c)$ equals the distance to a fixed hyperplane, hence is a linear function with slope one. It follows that $\mu = 0$ in such a neighbourhood. This shows that $\operatorname{supp} \mu \subset \overline{R}$, where R denotes the ridge.

It is easy to see that actually $\operatorname{supp} \mu = \overline{R}$. Therefore, if $x \in \Omega \setminus \operatorname{supp} \mu$ then x has exactly one closest point y on $\partial\Omega$. Clearly any point on the segment from x to y also has y as the unique closest point on $\partial\Omega$, hence all this segment is in $\Omega \setminus R = \Omega \setminus \operatorname{supp} \mu$. From this $(M5)$ follows, and the other axioms are immediate to verify.

To show that the above constructed μ is the *unique* mother body, only axioms $(M1)$, $(M4)$ and $(M5)$ are needed. So assume ν is any measure (even a signed measure) satisfying these conditions and we shall show that $\nu = \mu$. Set

$$v = U^\nu - U^\Omega.$$

Using the definition of U^ν and Fubini's theorem one easily verifies that both v itself and its gradient ∇v are locally integrable in $\mathbb{R}^n$.

We can write $\Omega = \cap_{j=1}^m H_j$ where the H_j are open half-spaces and m is minimal. By $(M5)$ for ν, each point $x \in \Omega \setminus \operatorname{supp} \nu$ can be connected with $\mathbb{R}^n \setminus \overline{\Omega}$ via a curve in $\mathbb{R}^n \setminus \operatorname{supp} \nu$. In a neighbourhood of such a curve $\Delta v = \chi_\Omega$. If the curve enters Ω through for example ∂H_j (only) it follows that $v = \frac{1}{2} d(\cdot, H_j^c)^2$ holds in the beginning (when the curve enters Ω), and then it automatically holds throughout the curve. In conclusion, for every $x \in \Omega \setminus \operatorname{supp} \nu$,

$$v(x) = \frac{1}{2} d(x, H_j^c)^2 \text{ for some } j. \tag{7.6}$$

Taking the gradient gives, with $C = \operatorname{diam}(\Omega)$,

$$|\nabla v(x)| \leq C < \infty$$

for all $x \in \Omega \setminus \operatorname{supp} \nu$, hence a.e. in Ω.

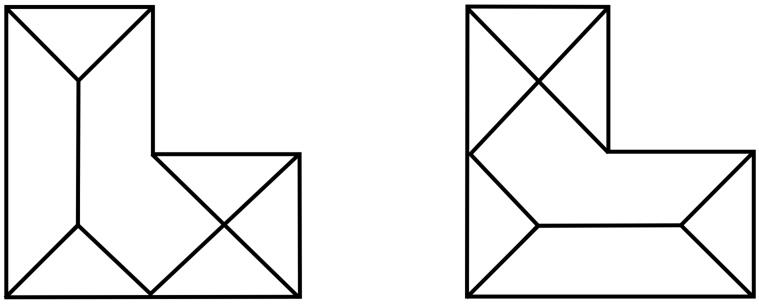

Figure 10: Zidarov's counterexample.

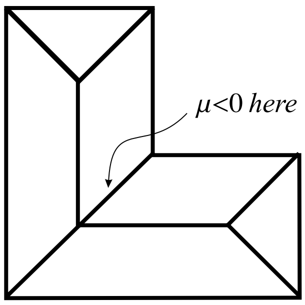

Figure 11: Axiom $(M3)$ violated.

Now, since ∇v is locally integrable its almost everywhere derivative represents its distributional derivative. Hence $\nabla v \in L^\infty$ in the sense of distributions, i.e., v is Lipschitz continuous. From this it follows that v in (7.6) can change representation between two j only on R. This easily leads to the conclusion that $v = u$ and hence that $\nu = \mu$, as desired.

Example 4. For nonconvex polyhedra in two dimensions mother bodies still exist [47] but they are not unique. See Figure 10 for a counterexample due to D. Zidarov [119].

The two mother bodies in Figure 10 are obtained by decomposing the polyhedron into two convex polyhedra (a square and a rectangle) in two different ways and using the unique mother bodies for the convex polyhedra. We see that the result does not respect the symmetry of the polyhedron along the diagonal through the nonconvex corner. If we allow signed measures, violating $(M3)$, we can however find a skeleton which respects this symmetry, and which looks more natural in general, see Figure 11. In this case μ is negative inside the nonconvex corner.

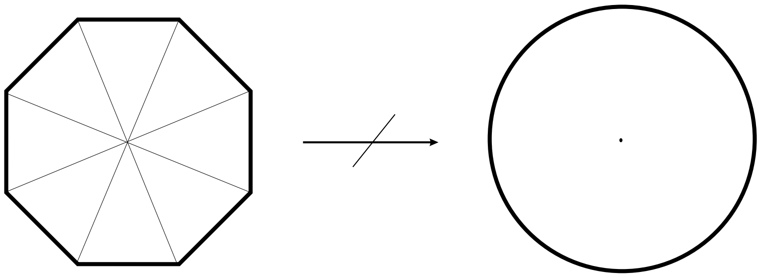

Figure 12: No convergence of mother bodies.

In general, a skeleton as the above one (violating $(M3)$, but respecting the geometry better) can be constructed for any polyhedron in two dimensions [109]. The situation for nonconvex polyhedra in higher dimensions is not fully clear at present. One thing which is known is that there are at most finitely many signed measures μ satisfying $(M1)$, $(M4)$ and $(M5)$, and that all these also satisfy $(M2)$, see [47].

Let us next give an example of instability of mother bodies.

EXAMPLE 5. For $\Omega = B(0,1) \subset \mathbb{R}^2$, the unit disc, $\mu = \pi\delta$ is the unique mother body (Example 1). Approximate this Ω by a regular polygon Ω_m with m corners on the unit circle. According to Example 3 its unique mother body μ_m sits on the ridge of Ω_m, the ridge in this case consisting of the m radii ("spokes") from the center of Ω_m to the corners. See Figure 12.

Setting $u_m = U^{\mu_m} - U^{\Omega_m}$, $d_m = d(\cdot, \Omega_m^c)$ we have $u_m = \frac{1}{2}d_m^2$ and

$$\nabla u_m = d_m \nabla d_m.$$

The density of μ_m on each of the m spokes equals the jump of the normal derivative of u_m across it, hence is proportional to d_m (note that $|\nabla d_m| = 1$). The constant of proportionality is obtained from $\int d\mu_m = |\Omega_m|$ and one ends up with

$$d\mu_m = 2\sin\frac{\pi}{m} \cdot (1-r)dr$$

on each of the radii in the ridge, r denoting the distance from the center.

Now the point with this example is that

$$\mu_m \not\to \mu \text{ as } m \to \infty,$$

despite $\Omega_m \to \Omega$ in many natural topologies. Indeed, since the ridge consists of m radii distributed over an angular opening of 2π we have, in polar coordinates and relating μ_m to area measure,

$$d\mu_m \approx 2\sin\frac{\pi}{m} \cdot (1-r)dr \cdot \frac{m}{2\pi}d\theta \rightharpoonup (\frac{1}{r} - 1)rdrd\theta$$

as $m \to \infty$. The limit measure here differs quite a lot from $\mu = \pi\delta$. For example, one-quarter of its mass is outside the disc of radius one-half.

Returning to the general discussion of mother bodies, the following proposition shows that the axioms $(M1)$-$(M5)$ are strong enough to guarantee several natural optimality properties (cf. [1], [2], [67], [74]).

Proposition 7.1. *Let μ be a mother body for Ω. Then among measures ν, ν_j satisfying $(M1)$ and $(M3)$ we have that*

(i) $\operatorname{supp}\mu$ *is* minimal *as a set: if* $\operatorname{supp}\nu \subset \operatorname{supp}\mu$ *then* $\nu = \mu$.

(ii) μ *is* maximal *with respect to* $\preccurlyeq$ *(see Section 4): if* $\mu \preccurlyeq \nu$ *then* $\nu = \mu$.

(iii) μ *is* extremal *as a point on the convex set defined by* $(M1)$, $(M3)$*: if* $\mu = \frac{1}{2}(\nu_1 + \nu_2)$ *then* $\mu = \nu_1 = \nu_2$.

Proof. It is enough to prove that $U^\nu = U^\mu$ (or $U^{\nu_j} = U^\mu$, in case (iii)) in $\mathbb{R}^n \setminus \operatorname{supp}\mu$ (hence a.e. in $\mathbb{R}^n$). Let D be a component of $\mathbb{R}^n \setminus \operatorname{supp}\mu$, set $w = U^\nu - U^\mu$ and we shall show that $w = 0$ in D. We have $w = 0$ in $D \setminus \overline{\Omega} \neq \varnothing$ by $(M1)$ and $(M5)$.

In case (i) we get $w = 0$ in all D by harmonic continuation.

In case (ii) we have $w \geq 0$ and $-\Delta w \geq 0$ in D. Hence either $w > 0$ in all D or $w = 0$ in all D. But the first alternative has already been excluded, so we again get $w = 0$ in D.

In case (iii) we have $\operatorname{supp} \frac{1}{2}(\nu_1 + \nu_2) = \operatorname{supp}\nu_1 \cup \operatorname{supp}\nu_2$. Hence $\operatorname{supp}\nu_j \subset \operatorname{supp}\mu$, and we are back to case (i). □

Although there is no strict uniqueness of mother bodies in general, the axioms are at least sufficient to prevent continuous deformations. This is (somewhat vaguely) expressed in the following proposition.

Proposition 7.2. *Assume $t \mapsto \mu(t)$ is a smoothly moving family of mother bodies for Ω. Then $\dot{\mu}(t) = 0$, i.e., $\mu(t) = \mu(0)$ for all t (dot denotes derivative with respect to t).*

Proof. By "smoothly moving" we mean that $\mu(t)$ moves in a smooth vector field $\xi = \xi(x,t)$ in Ω. Then

$$\dot{\mu}(t) + \operatorname{div}\,(\mu(t)\xi) = 0$$

(continuity equation, or balance of mass), by which

$$\dot{\mu}(t) \text{ is a distribution of order at most one,} \tag{7.7}$$

$$\operatorname{supp}\dot{\mu}(t) \subset \operatorname{supp}\mu(t). \tag{7.8}$$

We have $U^{\mu(t)} = U^\Omega$ outside Ω by $(M1)$, hence

$$U^{\dot{\mu}(t)} = 0 \text{ outside } \Omega.$$

Therefore, by harmonic continuation, using (7.8) and $(M5)$,

$$U^{\dot{\mu}(t)} = 0 \text{ in } \mathbb{R}^n \setminus \operatorname{supp}\mu(t),$$

hence a.e. in $\mathbb{R}^n$. It follows from (7.7) that $U^{\dot{\mu}(t)} \in L^1_{loc}$, hence we conclude that $U^{\dot{\mu}(t)} = 0$ as a distribution, so that $\dot{\mu}(t) = -\Delta U^{\dot{\mu}(t)} = 0$. □

A way to ensure strict uniqueness of mother bodies is to impose something which is stronger than $(M5)$. One example is

$$(M6) \qquad \operatorname{supp}\mu \text{ does not disconnect any open set.}$$

This means that for any open set D, $D \setminus \operatorname{supp}\mu$ is connected whenever D is. Clearly, $(M6)$ implies $(M5)$. We have

Proposition 7.3. *If μ and ν are two mother bodies for Ω and one of them satisfies $(M6)$, then $\nu = \mu$.*

Proof. The proof follows that of Proposition 7.1: keeping the notations from that proof, if ν satisfies $(M6)$ then $\operatorname{supp}\nu$ does not disconnect D. Therefore $w = 0$ in $D \setminus \operatorname{supp}\nu$ by harmonic continuation, hence $w = 0$ a.e. in D, which is enough for the conclusion. □

One example in which $(M6)$ holds is when μ has support in a finite number of points. In this case it is immediate from Theorem 5.3 and Proposition 7.3 that μ is the unique mother body for the swept domain (or open set) $\Omega = \Omega(\mu)$. In the case of two dimensions the boundary of this Ω is an algebraic curve, as was remarked in Section 5. The question naturally arises whether every domain bounded by an algebraic curve has a mother body.

Assume to this end that we have a domain $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ is given by

$$P(z, \bar{z}) = 0 \qquad (z \in \partial\Omega),$$

where $P(z, w)$ is a polynomial (satisfying $P(z, w) = \overline{P(\bar{w}, \bar{z})}$ to ensure that $P(z, \bar{z})$ is realvalued). Solving $P(z, w) = 0$ for w gives an algebraic function which close to $\partial\Omega$ has a single-valued branch $w = S(z)$, called the **Schwarz function** of $\partial\Omega$ [21], [4], [107] and which is characterized by

$$S(z) = \bar{z} \text{ on } \partial\Omega.$$

Using the Schwarz function it is easy to perform at least one major step in the construction of a mother body, namely that of harmonic continuation of U^Ω down to a small set. For simplicity, let us just consider ∇U^Ω instead, which is essentially the same thing as the Cauchy transform $\hat{\chi}_\Omega$. For this we have, in the presence of an algebraic Schwarz function and for $z \notin \overline{\Omega}$,

$$\hat{\chi}_\Omega(z) = \frac{1}{\pi}\int_\Omega \frac{dm(\zeta)}{z-\zeta} = \frac{1}{2\pi i}\int_\Omega \frac{d\bar{\zeta}d\zeta}{z-\zeta}$$
$$= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\bar{\zeta}d\zeta}{z-\zeta} = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{S(\zeta)d\zeta}{z-\zeta}.$$

A graviequivalence of the weak form (5.6), i.e. $\hat{\chi}_\Omega = \hat{\mu}$ outside Ω in present notation, means that the above expression should be identified with

$$\hat{\mu}(z) = \frac{1}{\pi}\int \frac{d\mu(\zeta)}{z-\zeta}.$$

Now $S(z)$ being algebraic means that it has singularities in Ω only in form of poles and/or branch points. Therefore the contour $\partial\Omega$ of integration in $\int_{\partial\Omega} \frac{S(\zeta)d\zeta}{z-\zeta}$ can be deformed down to single points and suitable cuts between branch points, the cuts being chosen so that $S(z)$ becomes single-valued on the complement. The poles are uniquely determined, but the cuts are subject to certain degrees of freedom. However, the requirement $(M3)$ that $\mu \geq 0$, or just that μ shall be real-valued, singles out just a few possibilities for the cuts. Indeed, comparing the expressions for $\hat{\chi}_\Omega$ and $\hat{\mu}$ we see that in order to have μ real-valued, the cuts γ have to satisfy

$$\operatorname{Re}\left([S(z)]_\gamma dz\right) = 0 \text{ along } \gamma. \tag{7.9}$$

Here $[S(z)]_\gamma$ denotes the jump of $S(z)$ across γ.

Clearly (7.9) permits only finitely many directions of γ at each point $z \in \Omega$. All candidates of mother bodies for Ω are obtained from the poles of $S(z)$ and by filling in the finitely-valued direction fields in Ω defined by (7.9), to get the admissable cuts. It follows that Ω can have at most finitely many mother bodies. See [102] for further discussions and results, for example, analysis at the branch points of $S(z)$.

It may very well happen that none of the candidates obtained above really satisfies all of $(M1)$-$(M5)$, so a domain bounded by an algebraic curve need not have a mother body. The simplest example of this is the conformal image of the unit disc under a univalent polynomial of degree two.

We have so far discussed the case that $\hat{\mu}$ is a rational function and the more general case that the Schwarz function $S(z)$ is an algebraic function. What about the still more general case that $\hat{\mu}$ is an algebraic function?

We shall just indicate one recent result on this question. Let

$$Q_m(z) = (z - z_1) \dots (z - z_m)$$

be a monic polynomial of degree m. Then one branch of $Q_m(z)^{-1/m}$ behaves like $1/z$ at infinity, hence as the Cauchy transform of a measure. In [8] the authors show, among many other things, that there exists a unique measure μ (with compact support) satisfying

$$\hat{\mu}(z)^m = \frac{1}{Q(z)} \text{ a.e. in } \mathbb{C}$$

and that this measure in addition satisfies

(i) $\operatorname{supp}\mu$ is a finite union of smooth curves (which can be calculated explicitly),

(ii) $\{z_1, \dots, z_m\} \subset \operatorname{supp}\mu \subset \operatorname{conv}\{z_1, \dots, z_m\}$,

(iii) $\operatorname{supp}\mu$ and $\mathbb{C} \setminus \operatorname{supp}\mu$ are both connected.

Although there is no domain Ω present here we see that whenever we make partial balayage of μ, it will be a mother body for the domain $\Omega(\mu)$ obtained.

8. Experiment

In this section we show pictures of a real Hele-Shaw flow. The experiment is performed by pouring a blob of hair schampoo on a glass plate and then pressing slowly with another glass plate from above. Thus we consider the squeezing version of the Hele-Shaw moving boundary problem. Figure 13 below shows the performance of the experiment.

The well-posedness and stability of the process shows up by the fluid blob quickly becoming more regular and circular (this can be seen in pictures two to five in Figure 13). Next the process is reversed by separating the plates (the last pictures in Figure 13) while trying to keep them as parallel as possible. One may need to use a knife or similar to bend the plates apart. Then "fingers" on the boundary immediately start to develop, and eventually a skeleton pattern arises, which reminds quite a lot of the ridge structure in mother bodies for polyhedra. The more detailed pictures in Figures 14 and 15 show this clearly.

This experiment was first shown to me by Sam Howison at a conference in Stockholm 1989. Pictures similar to those in Figures 13-15 can be found in many papers and books. One example is the popular science book [81] (p.194 there).

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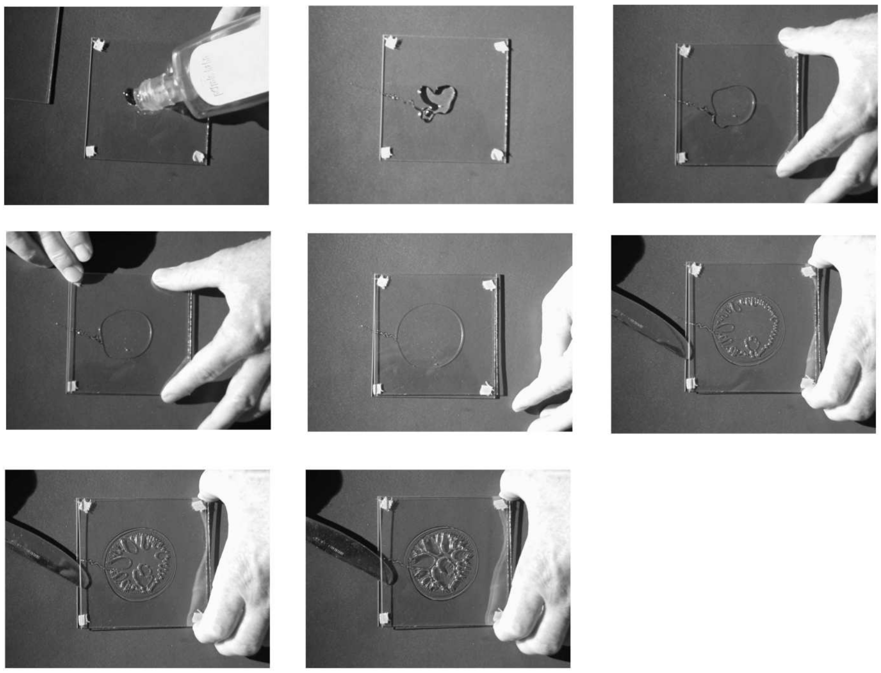

Figure 13: Performance of Hele-Shaw experiment.

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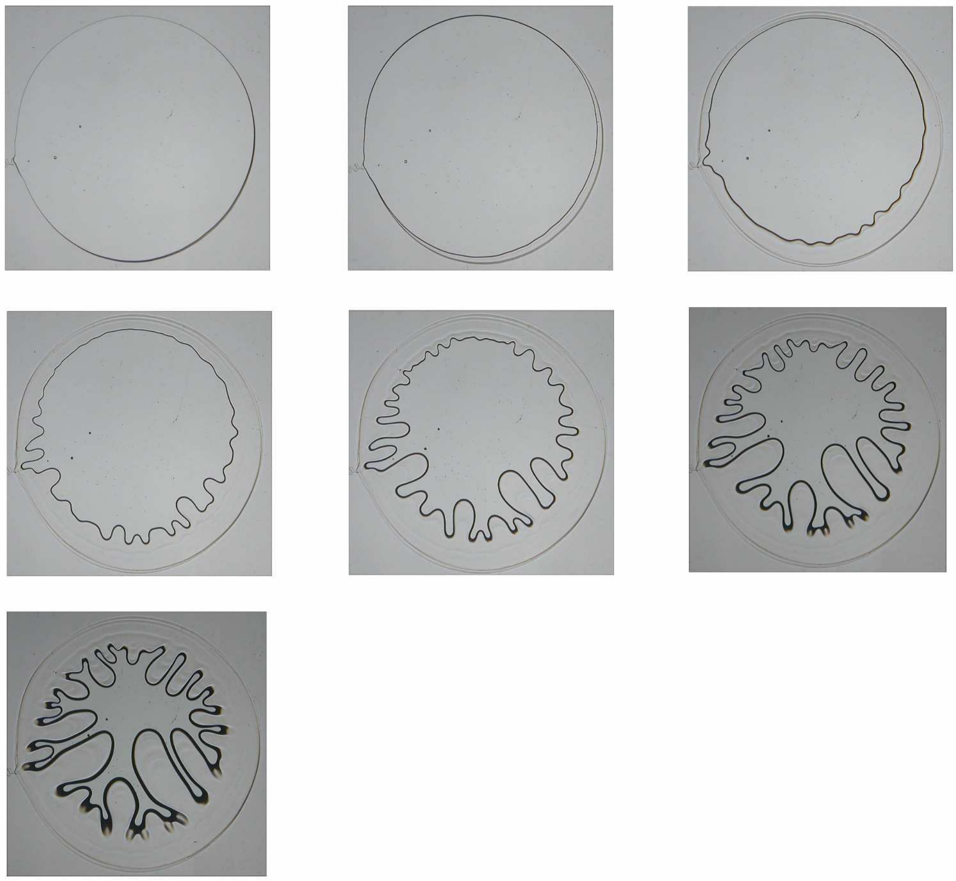

Figure 14: Dynamics in the ill-posed direction.

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Figure 15: Ending up with a skeleton.

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